SECTION FIVE
STRUCTURAL THEORY

Akbar Tamboli, Michael Xing, Mohsin Ahmed
Thornton-Tomasetti Engineers, Newark, New Jersey

STRUCTURAL THEORY CREATES IDEALIZATION OF STRUCTURE FOR PURPOSES OF ANALYSIS

Structural modeling is an essential and important tool in structural engineering. Over the past 200 years, many of the most significant contributions to the understanding of the structures have been made by Scientist Engineers while working on mathematical models, which were used for real structures.

Application of mathematical model of any sort to any real structural system must be idealized in some fashion; that is, an analytical model must be developed. There has never been an analytical model, which is a precise representation of the physical system. While the performance of the structure is the result of natural effects, the development and thus the performance of the model is entirely under the control of the analyst. The validity of the results obtained from applying mathematical theory to the study of the model therefore rests on the accuracy of the model. While this is true, it does not mean that all analytical models must be elaborate, conceptually sophisticated devices. In some cases very simple models give surprisingly accurate results. While in some other cases they may yield answers, which deviate markedly from the true physical behavior of the model, yet be completely satisfactory for the problem at hand.

Structure design is the application of structural theory to ensure that buildings and other structures are built to support all loads and resist all constraining forces that may be reasonably expected to be imposed on them during their expected service life, without hazard to occupants or users and preferably without dangerous deformations, excessive sideways (drift), or annoying vibrations. In addition, good design requires that this objective be achieved economically.

Provision should be made in application of structural theory to design for abnormal as well as normal service conditions. Abnormal conditions may arise as a result of accidents, fire, explosions, tornadoes, severer-than-anticipated earthquakes, floods, and inadvertent or even deliberate overloading of building components. Under such conditions, parts of a building may be damaged. The structural system, however, should be so designed that the damage will be limited in extent and undamaged portions of the building will remain stable. For the purpose, structural elements should be proportioned and arranged to form a stable system under normal service conditions. In addition, the system should have sufficient continuity and ductility, or energy-absorption capacity, so that if any small portion of it should sustain damage, other parts will transfer loads (at least until repairs can be made) to remaining structural components capable of transmitting the loads to the ground.

(''Steel Design Handbook, LRFD Method'', Akbar R. Tamboli Ed., McGraw-Hill 1997. ''Design Methods for Reducing the Risk of Progressive Collapse in Buildings''. NBS Buildings Science Series 98, National Institute of Standards and Technology, 1997. ''Handbook of Structural Steel Connection Design and Details'', Akbar R. Tamboli Ed., McGraw-Hill 1999'').

5.1 DESIGN LOADS

Loads are the external forces acting on a structure. Stresses are the internal forces that resist them. Depending on that manner in which the loads are applied, they tend to deform the structure and its components—tensile forces tend to stretch, compressive forces to squeeze together, torsional forces to twist, and shearing forces to slide parts of the structure past each other.

5.1.1 Types of Loads

External loads on a structure may be classified in several different ways. In one classification, they may be considered as static or dynamic.

Static loads are forces that are applied slowly and then remain nearly constant. One example is the weight, or dead load, of a floor or roof system.

Dynamic loads vary with time. They include repeated and impact loads.

Repeated loads are forces that are applied a number of times, causing a variation in the magnitude, and sometimes also in the sense, of the internal forces. A good example is an off-balance motor.

Impact loads are forces that require a structure or its components to absorb energy in a short interval of time. An example is the dropping of a heavy weight on a floor slab, or the shock wave from an explosion striking the walls and roof of a building.

External forces may also be classified as distributed and concentrated.

Uniformly distributed loads are forces that are, or for practical purposes may be considered, constant over a surface area of the supporting member. Dead weight of a rolled-steel I beam is a good example.

Concentrated loads are forces that have such a small contact area as to be negligible compared with the entire surface area of the supporting member. A beam supported on a girder, for example, may be considered, for all practical purposes, a concentrated load on the girder.

Another common classification for external forces labels them axial, eccentric, and torsional.

An **axial load** is a force whose resultant passes through the centroid of a section under consideration and is perpendicular to the plane of the section.

An **eccentric load** is a force perpendicular to the plane of the section under consideration but not passing through the centroid of the section, thus bending the supporting member (see Arts. 5.4.2, 5.5.17, and 5.5.19).

Torsional loads are forces that are offset from the shear center of the section under consideration and are inclined to or in the plane of the section, thus twisting the supporting member (see Arts. 5.4.2 and 5.5.19).

Also, building codes classify loads in accordance with the nature of the source. For example:

Dead loads include materials, equipment, constructions, or other elements of weight supported in, on, or by a building, including its own weight, that are intended to remain permanently in place.

Live loads include all occupants, materials, equipment, constructions, or other elements of weight supported in, on, or by a building and that will or are likely to be moved or relocated during the expected life of the building.

Impact loads are a fraction of the live loads used to account for additional stresses and deflections resulting from movement of the live loads.

Wind loads are maximum forces that may be applied to a building by wind in a mean recurrence interval, or a set of forces that will produce equivalent stresses.

Snow loads are maximum forces that may be applied by snow accumulation in a mean recurrence interval.

Seismic loads are forces that produce maximum stresses or deformations in a building during an earthquake.

5.1.2 Service Loads

In designing structural members, designers should use whichever is larger of the following:

1. Loadings specified in the local or state building code.
2. Probable maximum loads, based not only on current site conditions and original usage of proposed building spaces but also on possible future events. Loads that are of uncertain magnitude and that may be treated as statistical variables should be selected in accordance with a specific probability that the chosen magnitudes will not be exceeded during the life of the building or in accordance with the corresponding mean recurrence interval. The mean recurrence interval generally used for ordinary permanent buildings is 50 years. The interval, however, may be set at 25 years for structures with no occupants or offering negligible risk to life, or at 100 years for permanent buildings with a high degree of sensitivity to the loads and an unusually high degree of hazard to life and property in case of failure.

In the absence of a local or state building code, designers can be guided by loads specified in a national model building code or by the following data:

Loads applied to structural members may consist of the following, alone or in combination: dead, live, impact, earth pressure, hydrostatic pressure, snow, ice, rain, wind, or earthquake loads; constraining forces, such as those resulting from restriction of thermal, shrinkage, or moisture-change movements; or forces caused by displacements or deformations of members, such as those caused by creep, plastic flow, differential settlement, or sideways (drift).

Dead Loads. Actual weights of materials and installed equipment should be used. See Tables 5.1 and 5.2*c*.

TABLE 5.1 Minimum Design Dead Loads

Walls	lb/ft^2
Clay brick	
High-absorption, per 4-in wythe	34
Medium-absorption, per 4-in wythe	39
Low-absorption, per 4-in wythe	46
Sand-lime brick, per 4-in wythe	38
Concrete brick	
4-in, with heavy aggregate	46
4-in, with light aggregate	33
Concrete block, hollow	
8-in, with heavy aggregate	55
8-in, with light aggregate	35
12-in, with heavy aggregate	85
12-in, with light aggregate	55
Clay tile, loadbearing	
4-in	24
8-in	42
12-in	58
Clay tile, nonloadbearing	
2-in	11
4-in	18
8-in	34
Furring tile	
1½-in	8
2-in	10
Glass block, 4-in	18
Gypsum block, hollow	
2-in	9.5
4-in	12.5
6-in	18.5

Floor Finishes	lb/ft^2
Asphalt block, 2-in	24
Cement, 1-in	12
Ceramic or quarry tile, 1-in	12
Hardwood flooring, ⅞-in	4
Plywood subflooring, ½-in	1.5
Resilient flooring, such as asphalt tile and linoleum	2
Slate, 1-in	15
Softwood subflooring, per in of thickness	3
Terrazzo, 1-in	13
Wood block, 3-in	4

Wood joists, double wood floor, joist size	lb/ft^2	
	12-in spacing	16-in spacing
2 × 6	6	5
2 × 8	6	6
2 × 10	7	6
2 × 12	8	7
3 × 6	7	6
3 × 8	8	7
3 × 10	9	8
3 × 12	11	9
3 × 14	12	10

Concrete Slabs	lb/ft^2
Stone aggregate, reinforced, per in of thickness	12.5
Slag, reinforced, per in of thickness	11.5
Lightweight aggregate, reinforced, per in of thickness	6 to 10

TABLE 5.1 Minimum Design Dead Loads (*Continued*)

Masonry	lb/ft^3
Cast-stone masonry	144
Concrete, stone aggregate, reinforced	150
Ashlar:	
Granite	165
Limestone, crystalline	165
Limestone, oölitic	135
Marble	173
Sandstone	144
Roof and Wall Coverings	lb/ft^2
Clay tile shingles	9 to 14
Asphalt shingles	2
Composition:	
3-ply ready roofing	1
4-ply felt and gravel	5.5
5-ply felt and gravel	6
Copper or tin	1
Corrugated steel	2
Sheathing (gypsum), ½-in	2
Sheathing (wood), per in thickness	3
Slate, ¼-in	10
Wood shingles	2
Waterproofing	lb/ft^2
Five-ply membrane	5
Ceilings	lb/ft^2
Plaster (on tile or concrete)	5
Suspended metal lath and gypsum plaster	10
Suspended metal lath and cement plaster	15
Suspended steel channel supports	2
Gypsumboard per ¼-in thickness	1.1
Floor Fill	lb/ft^2
Cinders, no cement, per in of thickness	5
Cinders, with cement, per in of thickness	9
Sand, per in of thickness	8
Partitions	lb/ft^2
Plaster on masonry	
Gypsum, with sand, per in of thickness	8.5
Gypsum, with lightweight aggregate, per in	4
Cement, with sand, per in of thickness	10
Cement, with lightweight aggregate, per in	5
Plaster, 2-in solid	20
Metal studs	
Plastered two sides	18
Gypsumboard each side	6
Wood studs, 2 × 4-in	
Unplastered	3
Plastered one side	11
Plastered two sides	19
Gypsumboard each side	7
Glass	lb/ft^2
Single-strength	1.2
Double-strength	1.6
Plate, ⅛-in	1.6
Insulation	lb/ft^2
Cork, per in of thickness	1.0
Foamed glass, per in of thickness	0.8
Glass-fiber bats, per in of thickness	0.06
Polystyrene, per in of thickness	0.2
Urethane	0.17
Vermiculite, loose fill, per in of thickness	0.5

TABLE 5.2 Minimum Design Live Loads

a. Uniformly distributed live loads, lb/ft^2, impact included[a]

Occupancy or use	Load	Occupancy or use	Load
Assembly spaces:		Marques	75
Auditoriums[b] with fixed seats	60	Morgue	125
Auditoriums[b] with movable seats	100	Office buildings:	
Ballrooms and dance halls	100	Corridors above first floor	80
Bowling alleys, poolrooms,		Files	125
similar recreational areas	75	Offices	50
Conference and card rooms	50	Penal institutions:	
Dining rooms, restaurants	100	Cell blocks	40
Drill rooms	150	Corridors	100
Grandstand and reviewing-stand		Residential:	
seating areas	100	Dormitories	
Gymnasiums	100	Nonpartitioned	60
Lobbies, first-floor	100	Partitioned	40
Roof gardens, terraces	100	Dwellings, multifamily:	
Skating rinks	100	Apartments	40
Stadium and arenas bleachers	100	Corridors	80
Bakeries	150	Hotels:	
Balconies (exterior)	100	Guest rooms, private cooridors	40
Up to 100 ft^2 on one- and two-		Public corridors	100
family houses	60	Housing, one- and two-family:	
Bowling alleys, alleys only	40	First floor	40
Broadcasting studios	100	Storage attics	80
Catwalks	40	Uninhabitable attics	20
Corridors:		Upper floors, habitable attics	30
Areas of public assembly, first-		Schools:	
floor lobbies	100	Classrooms	40
Other floors same as occupancy		Corridors above first floor	80
served, except as indicated		First floor corridors	100
elsewhere in this table		Shops with light equipment	60
Fire escapes:		Stairs and exitways	100
Single-family dwellings only	40	Handrails, vertical and horizontal	
Others	100	thrust, lb/lin ft	50
Garages:		Storage warehouse:	
Passenger cars	50	Heavy	250
Trucks and buses		Light	125
Hospitals:		Stores:	
Operating rooms, laboratories,		Retail:	
service areas	60	Basement and first floor	100
Patients' rooms, wards,		Upper floors	75
personnel areas	40	Wholesale	125
Corridors above first floor	80	Telephone equipment rooms	80
Kitchens other than domestic	150	Theaters:	
Laboratories, scientific	100	Aisles, corridors, lobbies	100
Libraries:		Dressing rooms	40
Corridors above first floor	80	Projection rooms	100
Reading rooms	60	Stage floors	150
Stack rooms, books and		Toilet areas	60
shelving at 65 lb/ft^3, but at			
least	150		
Manufacturing and repair areas:			
Heavy	250		
Light	125		

[a] See Eqs. (5.1) and (5.2).
[b] Including churches, schools, theaters, courthouses, and lecture halls.
[c] Use American Association of State Highway and Transportation Officials highway lane loadings.

TABLE 5.2 Minimum Design Live Loads (*Continued*)

b. Concentrated live loads[d]

Location	Load, lb
Elevator machine room grating (on 4-in^2 area)	300
Finish, light floor-plate construction (on 1-in^2 area)	200
Garages:	
Passenger cars:	
Manual parking (on 20-in^2 area)	2,000
Mechanical parking (no slab), per wheel	1,500
Trucks, buses (on 20-in^2 area), per wheel	16,000
Manufacturing	
Light	2,000
Heavy	3,000
Office floors (on area 2.5 ft square)	2,000
Scuttles, skylight ribs, and accessible ceilings (on area 2.5 ft square)	200
Sidewalks (on area 2.5 ft square)	8,000
Stair treads (on 4-in^2 area at center of tread)	300
Libraries (on area 2.5 ft square)	1,500
Hospitals (on area 2.5 ft square)	1,000
Schools (on area 2.5 ft square)	1,000
Stores (on area 2.5 ft square)	3,000

[d] Use instead of uniformly distributed live load, except for roof trusses, if concentrated loads produce greater stresses or deflections. Add impact factor for machinery and moving loads: 100% for elevators, 20% for light machines, 50% for reciprocating machines, 33% for floor or balcony hangers. For craneways, and a vertical force equal to 25% of maximum wheel load; a lateral force equal to 10% of the weight of trolley and lifted load, at the top of each rail; and a longitudinal force equal to 10% of maximum wheel loads, acting at top of rail.

Live Loads. These may be concentrated or distributed loads and should be considered placed on the building to produce maximum effects on the structural member being designed. Minimum live loads to be used in building design are listed in Table 5.2. These include an allowance for impact, except as noted in the footnote of Table 5.2*b*.

Partitions generally are considered to be live loads, because they may be installed at any time, almost anywhere, to subdivide interior spaces, or may be shifted from original places to other places in the future. Consequently, unless a floor is designed for a large live load, for example, 80 lb/ft^2, the weight of partitions should be added to other live loads, whether or not partitions are shown on the working drawings for building construction.

Because of the low probability that a large floor area contributing load to a specific structural member will be completely loaded with maximum design live loads, building codes generally permit these loads to be reduced for certain types of occupancy. Usually, however, codes do not permit any reduction for places of public assembly, dwellings, garages for trucks and buses, or one-way slabs. For areas with a minimum required live load exceeding 100 lb/ft^2 and for passenger-car garages, live loads on columns supporting more than one floor may be decreased 20%. Except for the preceding cases, a reduced live load L, lb/ft^2, may be computed from

TABLE 5.2 Minimum Design Live Loads (*Continued*)

c. Minimum design loads for materials

Material	Load, lb/ft^3	Material	Load, lb/ft^3
Aluminum, cast	165	Gravel, dry	104
Bituminous products:		Gypspum, loose	70
Asphalt	81	Ice	57.2
Petroleum, gasoline	42	Iron, cast	450
Pitch	69	Lead	710
Tar	75	Lime, hydrated, loose	32
Brass, cast	534	Lime, hydrated, compacted	45
Bronze, 8 to 14% tin	509	Magnesium alloys	112
Cement, portland, loose	90	Mortar, hardened;	
Cement, portland, set	183	Cement	130
Cinders, dry, in bulk	45	Lime	110
Coal, anthracite, piled	52	Riprap (not submerged):	
Coal, bituminous or lignite, piled	47	Limestone	83
Coal, peat, dry, piled	23	Sandstone	90
Charcoal	12	Sand, clean and dry	90
Copper	556	Sand, river, dry	106
Earth (not submerged):		Silver	656
Clay, dry	63	Steel	490
Clay, damp	110	Stone, ashlar:	
Clay and gravel, dry	100	Basalt, granite, gneiss	165
Silt, moist, loose	78	Limestone, marble, quartz	160
Silt, moist, packed	96	Sandstone	140
Sand and gravel, dry, loose	100	Shale, slate	155
Sand and gravel, dry, packed	110	Tin, cast	459
Sand and gravel, wet	120	Water, fresh	62.4
Gold, solid	1205	Water, sea	64

$$L = \left(0.25 + \frac{15}{\sqrt{A_I}}\right) L_o \tag{5.1}$$

where L_o = unreduced live load, lb/ft^2 (see Table 5.1*a*)
A_I = influence area, or floor area over which the influence surface for structural effects is significantly different from zero
= area of four surrounding bays for an interior column, plus similar area from supported floors above, if any
= area of two adjoining bays for an interior girder or for an edge column, plus similar areas from supported floors above, if any
= area of one bay for an edge girder or for a corner column, plus similar areas from supported floors above, if any

The reduced live load L, however, should not be less than $0.5L_o$ for members supporting one floor or $0.4L_o$ for members supporting two or ore floors.

Roofs used for promenades should be designed for a minimum life load of 60 lb/ft^2, and those used for gardens or assembly, for 100 lb/ft^2. Ordinary roofs should be designed for a minimum live load L, lb/ft^2, computed from

$$L = 20R_1R_2 \geq 12 \tag{5.2}$$

where R_1 = $1.2 - 0.001A_t$ but not less than 0.6 or more than 1.0
A_t = tributary area, ft^2, for structural member being designed
R_2 = $1.2 - 0.05r$ but not less than 0.6 or more than 1.0
r = rise of roof in 12 in for a pitched roof or 32 times the ratio of rise to span for an arch or dome

This minimum live load need not be combined with snow load for design of a roof but should be designed for the larger of the two.

Subgrade Pressures. Walls below grade should be designed for lateral soil pressures and the hydrostatic pressure of subgrade water, plus the load from surcharges at ground level. Design pressures should take into account the reduced weight of soil because of buoyancy when water is present. In design of floors at or below grade, uplift due to hydrostatic pressures on the underside should be considered.

Wind Loads. Horizontal pressures produced by wind are assumed to act normal to the faces of buildings for design purposes and may be directed toward the interior of the buildings or outward (Arts. 3.2.1 and 3.2.2). These forces are called velocity pressures because they are primarily a function of the velocity of the wind striking the buildings. Building codes usually permit wind pressures to be either calculated or determined by tests on models of buildings and terrain if the tests meet specified requirements (see Art. 3.2.2). Codes also specify procedures for calculating wind loads, such as the following:

Velocity pressures due to wind to be used in building design vary with type of terrain, distance above ground level, importance of building, likelihood of hurricanes, and basic wind speed recorded near the building site. The wind pressures are assumed to act normal to the building facades.

The basic wind speed used in design is the fastest-mile wind speed recorded at a height of 10 m (32.8 ft) above open, level terrain with a 50-year mean recurrence interval.

Unusual wind conditions often occur over rough terrain and around ocean promontories. Basic wind speeds applicable to such regions should be selected with the aid of meteorologists and the application of extreme-value statistical analysis to anemometer readings taken at or near the site of the proposed building. Generally, however, minimum basic wind velocities are specified in local building codes and in national model building codes but should be used with discretion, because actual velocities at a specific sites and on a specific building may be significantly larger. In the absence of code specifications and reliable data, basic wind speed at a height of 10 m above grade may be approximated for preliminary design from the following:

Coastal areas, northwestern and southeastern United States and mountainous area	110 mph
Northern and central United States	90 mph
Other parts of the contiguous states	80 mph

For design purposes, wind pressures should be determined in accordance with the degree to which terrain surrounding the proposed building exposes it to the wind. Exposures may be classified as follows:

Exposure A applies to centers of large cities, where for at least one-half mile upwind from the building the majority of structures are over 70 ft high and lower buildings extend at least one more mile upwind.

Exposure B applies to wooded or suburban terrain or to urban areas with closely spaced buildings mostly less than 70 ft high, where such conditions prevail upwind for a distance from the building of at least 1500 ft or 10 times the building height.

Exposure C exists for flat, open country or exposed terrain with obstructions less than 30 ft high.

Exposure D applies to flat unobstructed areas exposed to wind blowing over a large expanse of water with a shoreline at a distance from the building or not more than 1500 ft or 10 times the building height.

For design purposes also, the following formulas may be used to determine, for heights z (in feet) greater than 15 ft above ground, a pressure coefficient K for converting wind speeds to pressures.

For Exposure A, for heights up to 1500 ft above ground level,

$$K = 0.000517 \left(\frac{z}{32.8}\right)^{2/3} \tag{5.3}$$

For z less than 15 ft, $K = 0.00031$.

For Exposure B, for heights up to 1200 ft above ground level,

$$K = 0.00133 \left(\frac{z}{32.8}\right)^{4/9} \tag{5.4}$$

For z less than 15 ft, $K = 0.00095$.

For Exposure C, for heights up to 900 ft above ground level,

$$K = 0.00256 \left(\frac{z}{32.8}\right)^{2/7} \tag{5.5}$$

For z less than 15 ft, $K = 0.0020$.

For Exposure D, for heights up to 700 ft above ground level,

$$K = 0.00357 \left(\frac{z}{32.8}\right)^{1/5} \tag{5.6}$$

For z less than 15 ft, $K = 0.0031$.

For ordinary buildings not subject to hurricanes, the velocity pressure q_z, psf, at height z may be calculated from

$$q_z = KV^2 \tag{5.7}$$

where V = basic wind speed, mi/hr, but not less than 70 mi/hr.

For important buildings, such as hospitals and communication buildings, for buildings sensitive to wind, such as slender skyscrapers, and for buildings presenting a high degree of hazard to life and property, such as auditoriums, q_z computed from Eq. (5.7) should be increased 15%.

To allow for hurricanes, q_z should be increased 10% for ordinary buildings and 20% for important, wind-sensitive or high-risk buildings along coastlines. These increases may be assumed to reduce uniformly with distance from the shore to zero for ordinary buildings and 15% for the more important or sensitive buildings at points 100 mi inland.

Wind pressures on low buildings are different at a specific elevation from those on tall buildings. Hence, building codes may give different formulas for pressures for the two types of construction. In any case, however, design wind pressure should be a minimum of 10 psf.

Multistory Buildings. For design of the main wind-force resisting system of ordinary, rectangular, multistory buildings, the design pressure at any height z, ft, above ground may be computed from

$$p_{zw} = G_o C_{pw} q_z \tag{5.8}$$

where p_{zw} = design wind pressure, psf, on windward wall
G_o = gust response factor
C_{pw} = external pressure coefficient
q_z = velocity pressure computed from Eq. (5.7) and modified for hurricanes and building importance, risks, and wind sensitivity

For windward walls, C_{pw} may be taken as 0.8. For side walls, C_{pw} may be assumed as -0.7 (suction). For roofs and leeward walls, the design pressure at elevation z is

$$p_{zl} = G_o C_p q_h \tag{5.9}$$

where p_{zl} = design pressure, psf, on roof or leeward wall
C_p = external pressure coefficient for roof or leeward wall
q_h = velocity pressure at mean roof height h (see Fig. 3.1d)

In these equations, the gust response factor may be taken approximately as

$$G_o = 0.65 + \frac{8.58D}{(h/30)^n} \geq 1 \tag{5.10}$$

where D = 0.16 for Exposure A, 0.10 for Exposure B, 0.07 for Exposure C, and 0.05 for Exposure D
n = ⅓ for Exposure A, $2/9$ for Exposure B, $1/7$ for Exposure C, and 0.1 for Exposure D
h = mean roof height, ft

For leeward walls, subjected to suction, C_p depends on the ratio of the depth d to width b of the building and may be assumed as follows:

d/b =	1 or less	2	4 or more
C_p =	-0.5	-0.3	-0.2

The negative sign indicates suction. Table 5.3 lists values of C_p for pressures on roofs.

Flexible Buildings. These are structures with a fundamental natural frequency less than 1 Hz or with a ratio of height to least horizontal dimension (measured at mid-height for buildings with tapers or setbacks) exceeding 5. For such buildings, the main wind-force resisting system should be designed for a pressure on windward walls at any height z, ft, above ground computed from

TABLE 5.3 External Pressure Coefficients C_p for Roofs*

Flat roofs	−0.7
Wind parallel to ridge of sloping roof	
h/b or $h/d \leq 2.5$	−0.7
h/b or $h/.d > 2.5$	−0.8
Wind perpendicular to ridge of sloping roof, at angle θ with horizontal	
Leeward side	−0.7
Windward side	

	Slope of roof θ, deg					
h/s	10	20	30	40	50	60 or more
0.3 or less	0.2	0.2	0.3	0.4	0.5	
0.5	−0.9	−0.75	−0.2	0.3	0.5	0.01θ
1.0	−0.9	−0.75	−0.2	0.3	0.5	
1.5 or more	−0.9	−0.9	−0.9	0.35	0.21	

* h = height of building, ft: d = depth, ft, of building in direction of wind: b = width, ft, of building transverse to wind.

Based on data in ANSI A58.1-1981.

$$p_{zw} = G_f C_{pw} q_z \tag{5.11}$$

where G_f = gust response factor determined by analysis of the system taking into account its dynamic properties. For leeward walls of flexible buildings,

$$p_{zl} = G_f C_p q_h \tag{5.12}$$

Requiring a knowledge of the fundamental frequency, structural damping characteristics, and type of exposure of the building, the formula for G_f is complicated, but computations may be simplified somewhat by use of tables and charts in the ASCE 7-98 standard.

One-Story Buildings. For design of the main wind-force resisting system of rectangular, one-story buildings, the design pressure at any height z, ft, above ground may be computed for windward walls from

$$p_{zw} = (G_o C_p + C_{pI}) q_z \tag{5.13}$$

where C_{p1} = 0.75 is the percentage of openings in one wall exceeds that of other walls by 10% or more
= 0.25 for all other cases

For roofs and leeward walls, the design pressure at elevation z is

$$p_{zl} = G_o C_p q_h - C_{p2} q_z \tag{5.14}$$

where C_{p2} = +0.75 or −0.25 if the percentage of openings in one wall exceeds that of other walls by 10% or more
= ±0.25 for all other cases

(Positive signs indicate pressures acting toward a wall; negative signs indicate pressures acting away from the wall.)

In ASCE-7-95 and 98, the basic wind speed changed from fast mile wind to 3-second gust wind speed in miles per hour. The wind speed values on the basic wind speed map has changed. This change should not have any big impact on the wind pressure. However, confusion is easily created because all the major building codes including the IBC 2000 are still using old basic wind speed map based on fast mile wind, and they repeatedly refer to ASCE-7 95 or 98. It is to be noted that the reference from the building codes to the ASCE-7 are either adoption of ASCE-7 as an alternative approach or for certain factors that are not related to the basic wind pressure.

In ASCE-7-95 and 98, new factors such as wind directionality factor, topographic factor were introduced, and gust effect factors were updated for rigid structures as well as for flexible/dynamically sensitive structures. The calculation became much more complicated than the approach in this book and the results should be more accurate. We suggest that for complicated structures it is necessary to use ASCE-7-98 method to check the results.

Snow, Ice, and Rain Loads. These, in effect, are nonuniformly distributed, vertical, live loads that are imposed by nature and hence are generally uncertain in magnitude and duration. They may occur alone or in combination. Design snow loads preferably should be determined for the site of the proposed building with the advice of meteorologists and application of extreme-value statistical analysis to rain and snow records for the locality.

Rain loads depend on drainage and may become large enough to cause roof failure when drainage is blocked (see Art. 3.4.3).

Ice loads are created when snow melts, then freezes, or when rain follows a snow storm and freezes. These loads should be considered in determining the design snow load. Snow loads may consist of pure snow or a mixture of snow, ice, and water.

Design snow loads on roofs may be assumed to be proportional to the maximum ground snow load p_g, lb/ft^2, measured in the vicinity of the building with a 50-year mean recurrence interval. Determination of the constant of proportionality should take into account:

1. Appropriate mean recurrence interval.

2. Roof exposure. Wind may blow snow off the roof or onto the roof from nearby higher roofs or create nonuniform distribution of snow.

3. Roof thermal conditions. Heat escaping through the roof melts the snow. If the water can drain off, the snow load decreases. Also, for sloped roofs, if they are warm, there is a tendency for snow to slide off. Insulated roofs, however, restrict heat loss from the interior and therefore are subjected to larger snow loads.

4. Type of occupancy and uses of building. More conservative loading should be used for public-assembly buildings, because of the risk of great loss of life and injury to occupants if overloads should cause the roof to collapse.

5. Roof slope. The steeper a roof, the greater is the likelihood of good drainage and that show will slide off.

In addition, roof design should take into account not only the design snow load uniformly distributed over the whole roof area but also possible unbalanced loading. Snow may be blown off part of the roof, and snow drifts may pile up over a portion of the roof.

For **flat roofs,** in the absence of building-code requirements, the basic snow load when the ground snow load p_g is 20 lb/ft^2 or less may be taken as

$$P_{\min} = p_g \tag{5.15}$$

When p_g is between 20 and 25 lb/ft^2, the minimum allowable design load is $p_{\min} =$ 20 lb/ft^2, and when p_g exceeds 25 lb/ft^2, the basic snow load may be taken as

$$p_f = 0.8p_g \tag{5.16}$$

where p_f = design snow load, lb/ft^2, for a flat roof that may have unheated space underneath and that may be located where the wind cannot be relied on to blow snow off, because of nearby higher structures or trees

p_g = ground snow load, lb/ft^2

For roofs sheltered from the wind, increase p_f computed from Eq. (5.16) by 20%, and for windy sites, reduce p_f 10%. For a poorly insulated roof with heated space underneath, decrease p_f by 30%.

Increase p_f 10% for large office buildings and public-assembly buildings, such as auditoriums, schools, factories. Increase p_f 20% for essential buildings, such as hospitals, communication buildings, police and fire stations, power plants, and for structures housing expensive objects or equipment. Decrease $p._f$ 20% for structures with low human occupancy, such as farm buildings.

The **ground snow load** p_g should be determined from an analysis of snow depths recorded at or near the site of the proposed building. For a rough estimate in the absence of building-code requirements, p_g may be taken as follows for the United States, except for mountainous regions:

0–5 lb/ft^2—southern states from about latitude N32° southward
10–15 lb/ft^2—Pacific coast between latitudes N32° and N40° and other states between latitudes N32° and N37°
20–30 lb/ft^2—Pacific coast from latitude N40° northward and other states between latitudes N37° and N40°
40–50 lb/ft^2—north Atlantic and central states between latitudes N40° and N43°
60–80 lb/ft^2—northern New England between latitudes N43° and N45° and central states from N43° northward
80–120 lb/ft^2—Maine above latitude N45°

For **sloping roofs,** the snow load depends on whether the roof will be warm or cold. In either case, the load may be assumed to be zero for roofs making an angle θ of 70° or more with the horizontal. Also, for any slope, the load need not be taken greater than p_f given by Eq. (5.16). For slopes θ, deg, between 0° and 70°, the snow load, lb/ft^2, acting vertically on the projection of the roof on a horizontal plane, may be computed for warm roofs from

$$p_s = \left(\frac{70 - \theta}{40}\right) p_f \le p_f \tag{5.17}$$

and for cold roofs from

$$p_s = \left(\frac{70 - \theta}{25}\right) p_f \le p_f \tag{5.18}$$

Hip and gable roofs should be designed for the condition of the whole roof loaded with p_s, and also with the windward wide unloaded and the leeward side carrying $1.5p_s$.

For **curved roofs,** the snow load on the portion that is steeper than 70p° may be taken as zero. For the less-steep portion, the load p_s may be computed as for a sloped roof, with θ taken as the angle with the horizontal of a line from the crown to points on the roof where the slope starts to exceed 70°. Curved roofs should be designed with the whole area fully loaded with p_s. They also should be designed for the case of snow only on the leeward side, with the load varying uniformly from $0.5p_s$ at the crown to $2p_s$ at points where the roof slope starts to exceed 30° and then decreasing to zero at points where the slope starts to exceed 70°.

Multiple folded-plate, sawtooth, and barrel-vault roofs similarly should be designed for unbalanced loads increasing from $0.5p_s$ at ridges to $3p_s$ in valleys.

Snow drifts may form on a roof near a higher roof that is less than 20 ft horizontally away. The reason for this is that wind may blow snow from the higher roof onto the lower roof. Drifts also may accumulate at projections above roofs, such as at parapets, solar collectors, and penthouse walls. Drift loads accordingly should be taken into account when:

1. The ground snow load p_g exceeds 10 lb/ft^2.
2. A higher roof exists (or may be built in the future) within 20 ft of the building, if the height differential, ft, exceeds $1.2p_f/\gamma$, where p_f is computed from Eq. (5.16) and γ is the snow density, lb/ft^3.
3. A projection extends a distance, ft, exceeding $1.2p_f/\gamma$ above the roof and is more than 15 ft long.

In computation of drift loads, the snow density γ, lb/ft^3, may be taken as follows:

p_g =	11–30	31–60	60 or more
γ =	15	20	25

The drift may be assumed to be a triangular prism with maximum height, located adjacent to a higher roof or along a projection, taken as $h_d = 2p_g/\gamma$, modified by factors for risk and exposure, described for flat roofs. Width of the prism should be at least 10 ft and may be taken as $3h_d$ for projections up to 50 ft long and as $4h_d$ for projections more than 50 ft long. Accordingly, the load varies uniformly with distance from a projection, from $h_d\gamma$ at the projection to zero. For drifts due to snow load from a higher roof at a horizontal distance S, fit, away horizontally ($S \leq 20$ ft), the maximum drift intensity may be taken as $h_d\gamma(20 - S)/20$.

Rain-Snow Load Combination. In roof design, account should be taken of the combination of the design snow load with a temporary water load from an intense rainstorm, including the effects of roof deflection on ponding. The added water load depends on the drainage characteristics of the roof, which, in turn, depend on the roof slope. For a flat roof, the rain surcharge may be taken as 8 lb/ft^2 for slopes less ¼ in/ft and as 5 lb/ft^2 for steeper slopes, except where the minimum allowable design snow load $p_{\min}$ exceeds p_f computed from Eq. (5.16). In such cases, these water surcharges may be reduced by $p_{\min} - p_f$.

(W. Tobiasson and R. Redfield, "Snow Loads for the United States," Part II, and S. C. Colbeck, "Snow Loads Resulting from Rain on Snow," U.S. Army Cold Regions Research and Engineering Laboratory, Hanover, N.H.)

Seismic Loads. These are the result of horizontal and vertical movements imposed on a building by earth vibrations during an earthquake. Changing accelerations of the building mass during the temblor create changing inertial forces. These are assumed in building design to act as seismic loads at the various floor and roof levels in proportion to the portion of the building mass at those levels. Because analysis of building response to such dynamic loading generally is very complex, building codes permit, for design of ordinary buildings, substitution of equivalent static loading for the dynamic loading (see Art. 5.18.6).

(''Minimum Design Loads for Buildings and Other Structures,'' ASCE 7-98, American Society of Civil Engineers, 345 E. 47th St., New York, NY 10164-0619; ''International Building Code 2000,'' 1998.)

5.1.3 Factored Loads

Structural members must be designed with sufficient capacity to sustain without excessive deformation or failure those combinations of service loads that will produce the most unfavorable effects. Also, the effects of such conditions as ponding of water on roofs, saturation of soils, settlement, and dimensional changes must be included. In determination of the structural capacity of a member or structure, a safety margin must be provided and the possibility of variations of material properties from assumed design values and of inexactness of capacity calculations must be taken into account.

Building codes may permit either of two methods, allowable-stress design or load–and–resistance factor design (also known as ultimate-strength design), to be used for a structural material. In both methods, design loads, which determine the required structural capacity, are calculated by multiplying combinations of service loads by factors. Different factors are applied to the various possible load combinations in accordance with the probability of occurrence of the loads.

In allowable-stress design, required capacity is usually determined by the load combination that causes severe cracking or excessive deformation. For the purpose, dead, live, wind, seismic, snow, and other loads that may be imposed simultaneously are added together, then multiplied by a factor equal to or less than 1. Load combinations usually considered in allowable-stress design are

(1) $D + L + (L_r \text{ or } S \text{ or } R)$

(2) $D + L + (W \text{ or } E/1.4)$

(3) $D + L + W + S/2$

(4) $D + L + S + W/2$

(5) $D + L + S + E/1.4$

(6) $0.9D - E/1.4$

where D = dead load
L = live loads due to intended use of occupancy, including partitions
L_r = roof live loads
S = snow loads
R = rain loads
W = wind loads
E = seismic loads

Building codes usually permit a smaller factor when the probability is small that combinations of extreme loads, such as dead load plus maximum live load plus maximum wind or seismic forces, will occur. Generally, for example, a factor of 0.75 is applied to load-combination sums (2) to (6). Such factors are equivalent to permitting higher allowable unit stresses for the applicable loading conditions than for load combination (1). The allowable stress is obtained by dividing the unit stress causing excessive deformation or failure by a factor greater than 1.

In load–and–resistance factor design, the various types of loads are each multiplied by a load factor, the value of which is selected in accordance with the probability of occurrence of each type of load. The factored loads are then added to obtain the total load a member or system must sustain. A structural member is selected to provide a load-carrying capacity exceeding that sum. This capacity is determined by multiplying the ultimate-load capacity by a resistance factor, the value of which reflects the reliability of the estimate of capacity. Load criteria generally used are as follows:

1. $1.4D$

2. $1.2D + 1.6L + 0.5(L_r \text{ or } S \text{ or } R)$

3. $1.2D + 1.6(L_r \text{ or } S \text{ or } R) + (0.5L \text{ or } 0.8W)$

4. $1.2D + 1.3W + 0.5\ (L_r \text{ or } S \text{ or } R)$

5. $1.2D + 1.0E + (0.5L \text{ or } 0.2S)$

6. $0.9D \pm (1.3W \text{ or } 1.0E)$

For garages, places of public assembly, and areas for which live loads exceed 100 lb/ft^2, the load factor usually is taken as unit for L in combinations 3, 4, and 5. For roof configurations that do not shed snow off the structure, the load factor should be taken as 0.7 for snow loads in combination 5.

For concrete structures where load combinations do not include seismic forces, the factored load combinations of ACI 318 Section 9.2 shall be used.

For both allowable stress design and strength design methods, elements and components shall be designed to resist the forces due to special seismic load combinations

a) $1.2D + 0.5L + E_m$

b) $0.9D - E_m$

For floors in places of public assembly, for live load in excess of 100 psf, and for parking garage live load, the load factor is taken as 1.0 for L. E_m is the maximum seismic effect of horizontal and vertical forces.

5.2 STRESS AND STRAIN

Structural capacity, or ultimate strength, is that property of a structural member that serves as a measure of is ability to support all potential loads without severe cracking or excessive deformations. To indicate when the limit on load-carrying usefulness has been reached, design specifications for the various structural materials establish allowable unit stresses or design strengths that may not be exceeded under

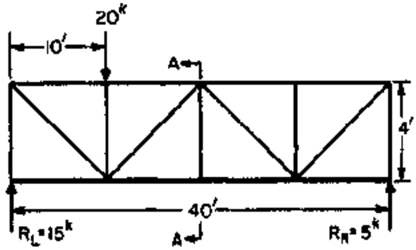


FIGURE 5.1 Truss in equilibrium under load. Upward acting forces equal those acting downward.

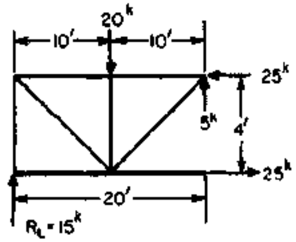


FIGURE 5.2 Portion of a truss is held in equilibrium by stresses in its components.

maximum loading. Structural theory provides methods for calculating unit stresses and for estimating deformations. Many of these methods are presented in the rest of this section.

5.2.1 Static Equilibrium

If a structure and its components are so supported that, after a very small deformation occurs, no further motion is possible, they are said to be in equilibrium. Under such circumstances, internal forces, or stresses, exactly counteract the loads.

Several useful conclusions may be drawn from the state of static equilibrium: Since there is no translatory motion, the sum of the external forces must be zero; and since there is no rotation, the sum of the moments of the external forces about any point must be zero.

For the same reason, if we consider any portion of the structure and the loads on it, the sum of the external and internal forces on the boundaries of that section must be zero. Also, the sum of the moments of these forces must be zero.

In Fig. 5.1, for example, the sum of the forces R_L and R_R needed to support the roof truss is equal to be the 20-kip load on the truss (1 kip = 1 kilopound = 1000 lb = 0.5 ton). Also, the sum of moments of the external forces is zero about any point. About the right end, for instance, it is $40 \times 15 - 30 \times 20 = 600 - 600$.

In Fig. 5.2 is shown the portion of the truss to the left of section *AA*. The internal forces at the cut members balance the external load and hold this piece of the truss in equilibrium.

Generally, it is convenient to decompose the forces acting on a structure into components parallel to a set of perpendicular axes that will simplify computations. For example, for forces in a single plane—a condition commonly encountered in building design—the most useful technique is to resolve all forces into horizontal and vertical components. Then, for a structure in equilibrium, if *H* represents the horizontal components, *V* the vertical components, and *M* the moments of the components about any point in the plane,

$$\Sigma H = 0 \qquad \Sigma V = 0 \quad \text{and} \quad \Sigma M = 0 \tag{5.19}$$

These three equations may be used to evaluate three unknowns in any nonconcurrent coplanar force system, such as the roof truss in Figs. 5.1 and 5.2. They may determine the magnitude of three forces for which the direction and point of application already are known, or the magnitude, direction, and point of application of a single force.

Suppose, for the truss in Fig. 5.1, the reactions at the supports are to be computed. Taking moments about the right end and equating to zero yields $40\ R_l - 30 \times 20 = 0$, from which left reaction $R_L = 600/40 = 15$ kips. Equating the sum of the vertical forces to zero gives $20 - 15 - R_R = 0$, from which the right reaction $R_R = 5$ kips.

5.2.2 Unit Stress and Strain

To ascertain whether a structural member has adequate load-carrying capacity, the designer generally has to compute the maximum unit stress produced by design loads in the member for each type of internal force—tensile, compressive, or shearing—and compare it with the corresponding allowable unit stress.

When the loading is such that the unit stress is constant over a section under consideration, the stress may be obtained by dividing the force by the area of the section. But in general, the unit stress varies from point to point. In that case, the unit stress at any point in the section is the limiting value of the ratio of the internal force on any small area to that area, as the area is taken smaller and smaller.

Sometimes in the design of a structure, unit stress may not be the prime consideration. The designer may be more interested in limiting the deformation or strain.

Deformation in any direction is the total change in the dimension of a member in that direction.

Unit strain in any direction is the deformation per unit of length in that direction.

When the loading is such that the unit strain is constant over a portion of a member, it may be obtained by dividing the deformation by the original length of that portion. In general, however, the unit strain varies from point to point in a member. Like a varying unit stress, it represents the limiting value of a ratio.

5.2.3 Hooke's Law

For many materials, unit strain is proportional to unit stress, until a certain stress, the **proportional limit,** is exceeded. Known as Hooke's law, this relationship may be written as

$$f = E\epsilon \quad \text{or} \quad \epsilon = \frac{f}{E} \tag{5.20}$$

where f = unit stress
ϵ = unit strain
E = modulus of elasticity

Hence, when the unit stress and modulus of elasticity of a material are known, the unit strain can be computed. Conversely, when the unit strain has been found, the unit stress can be calculated.

When a member is loaded and the unit stress does ot exceed the proportional limit, the member will return to its original dimensions when the load is removed. The **elastic limit** is the largest unit stress that can be developed without a permanent deformation remaining after removal of the load.

Some materials possess one or two **yield points.** These are unit stresses in the region of which there appears to be an increase in strain with no increase or a small decrease in stress. Thus, the materials exhibit plastic deformation. For materials that do not have a well-defined yield point, the offset yield strength is used as a measure of the beginning of plastic deformation.

The **offset yield strength,** or **proof stress** as it is sometimes referred to, is defined as the unit stress corresponding to a permanent deformation, usually 0.01% (0.0001 in/in) or 0.20% (0.002 in/in).

5.2.4 Constant Unit Stress

The simplest cases of stress and strain are those in which the unit stress and strain are constant. Stresses due to an axial tension or compression load or a centrally applied shearing force are examples; also an evenly applied bearing load. These loading conditions are illustrated in Figs. 5.3 to 5.6.

For the axial tension and compression loadings, we take a section normal to the centroidal axis (and to the applied forces). For the shearing load, the section is taken along a plane of sliding. And for the bearing load, it is chosen through the plane of contact between the two members.

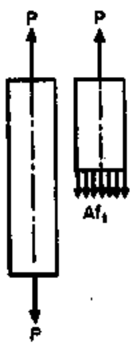


FIGURE 5.3 Tension member.

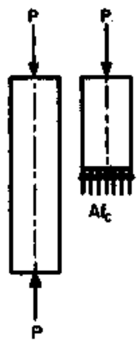


FIGURE 5.4 Compression member.

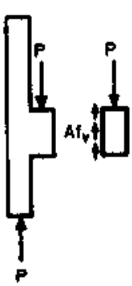


FIGURE 5.5 Bracket in shear.

FIGURE 5.6 Bearing load and pressure.

Since for these loading conditions, the unit stress is constant across the section, the equation of equilibrium may be written

$$P = Af \tag{5.21}$$

where P = load
f = a tensile, compressive, shearing, or bearing unit stress
A = cross-sectional area for tensile or compressive forces, or area on which sliding may occur for shearing forces, or contact area for bearing loads

For torsional stresses, see Art. 5.4.2.

The unit strain for the axial tensile and compressive loads is given by the equation

$$\epsilon = \frac{e}{L} \tag{5.22}$$

where ϵ = unit strain
e = total lengthening or shortening of the member
L = original length of the member

Applying Hooke's law and Eq. (5.22) to Eq. (5.21) yield a convenient formula for the deformation:

$$e = \frac{PL}{AE} \tag{5.23}$$

where P = load on the member
A = its cross-sectional area
E = modulus of elasticity of the material

[Since long compression members tend to buckle, Eqs. (5.21) to (5.23) are applicable only to short members.]

While tension and compression strains represent a simple stretching or shortening of a member, shearing strain represents a distortion due to a small rotation. The load on the small rectangular portion of the member in Fig. 5.5 tends to distort it into a parallelogram. The unit shearing strain is the change in the right angle, measured in radians.

Modulus of rigidity, or shearing modulus of elasticity, is defined by

$$G = \frac{v}{\gamma} \tag{5.24}$$

where G = modulus of rigidity
v = unit shearing stress
γ = unit shearing strain

It is related to the modulus of elasticity in tension and compression E by the equation

$$G = \frac{E}{2\,(1 + \mu)} \tag{5.25}$$

where μ is a constant known as Poisson's ratio.

5.2.5 Poisson's Ratio

Within the elastic limit, when a material is subjected to axial loads, it deforms not only longitudinally but also laterally. Under tension, the cross section of a member decreases, and under compression, it increases. The ratio of the unit lateral strain to the unit longitudinal strain is called Poisson's ratio.

For many materials, this ratio can be taken equal to 0.25. For structural steel, it is usually assumed to be 0.3.

Assume, for example, that a steel hanger with an area of 2 in^2 carries a 40-kip (40,000-lb) load. The unit stress is 40,000/2, or 20,000 psi. The unit tensile strain, taking the modulus of elasticity of the steel as 30,000,000 psi, is 20,000/30,000,000, or 0.00067 in/in. With Poisson's ratio as 0.3, the unit lateral strain is -0.3×0.00067, or a shortening of 0.00020 in/in.

5.2.6 Thermal Stresses

When the temperature of a body changes, its dimensions also change. Forces are required to prevent such dimensional changes, and stresses are set up in the body by these forces.

If α is the coefficient of expansion of the material and T the change in temperature, the unit strain in a bar restrained by external forces from expanding or contracting is

$$\epsilon = \alpha T \tag{5.26}$$

According to Hooke's law, the stress f in the bar is

$$f = E\alpha T \tag{5.27}$$

where E = modulus of elasticity.

5.2.7 Strain Energy

When a bar is stressed, energy is stored in it. If a bar supporting a load P undergoes a deformation e the energy stored in it is

$$U = \tfrac{1}{2}Pe \tag{5.28}$$

This equation assumes the load was applied gradually and the bar is not stressed beyond the proportional limit. It represents the area under the load-deformation curve up to the load P. Applying Eqs. (5.20) and (5.21) to Eq. (5.28) gives another useful equation for energy:

$$U = \frac{f^2}{2E} AL \tag{5.29}$$

where f = unit stress
E = modulus of elasticity of the material
A = cross-sectional area
L = length of the bar

Since AL is the volume of the bar, the term $f^2/2E$ indicates the energy stored per unit of volume. It represents the area under the stress-strain curve up to the stress f. Its value when the bar is stressed to the proportional limit is called the **modulus of resilience.** This modulus is a measure of the capacity of the material to absorb energy without danger of being permanently deformed and is of importance in designing members to resist energy loads.

Equation (5.28) is a general equation that holds true when the principle of superposition applies (the total deformation produced by a system of forces is equal to the sum of the elongations produced by each force). In the general sense, P in Eq. (5.28) represents any group of statically interdependent forces that can be completely defined by one symbol, and e is the corresponding deformation.

The strain-energy equation can be written as a function of either the load or the deformation.

For axial tension or compression:

$$U = \frac{P^2L}{2AE} \qquad U = \frac{AEe^2}{2L} \tag{5.30}$$

where P = axial load
e = total elongation not shortening
L = length of the member
A = cross-sectional area
E = modulus of elasticity

For pure shear:

$$U = \frac{V^2L}{2AG} \qquad U = \frac{AGe^2}{2L} \tag{5.31}$$

where V = shearing load
e = shearing deformation
L = length over which deformation takes place
A = shearing area
G = shearing modulus

For torsion:

$$U = \frac{T^2L}{2JG} \qquad U = \frac{JG\phi^2}{2L} \tag{5.32}$$

where T = torque
ϕ = angle of twist
L = length of shaft
J = polar moment of inertia of the cross section
G = shearing modulus

For pure bending (constant moment):

$$U = \frac{M^2L}{2EI} \qquad U = \frac{EI\theta^2}{2L} \tag{5.33}$$

where M = bending moment
θ = angle of rotation of one end of the beam with respect to the other
L = length of beam
I = moment of inertia of the cross section
E = modulus of elasticity

For beams carrying transverse loads, the strain energy is the sum of the energy for bending and that for shear.

See also Art. 5.10.4.

5.3 STRESSES AT A POINT

Tensile and compressive stresses are sometimes referred to also as **normal stresses,** because they act normal to the cross section. Under this concept, tensile stresses are considered as positive normal stresses and compressive stresses as negative.

5.3.1 Stress Notation

Suppose a member of a structure is acted upon by forces in all directions. For convenience, let us establish a reference set of perpendicular coordinate x, y, and z axes. Now let us take at some point in the member a small cube with sides parallel to the coordinate axes. The notations commonly used for the components of stress acting on the sides of this element and the directions assumed as positive are shown in Fig. 5.7.

For example, for the sides of the element perpendicular to the z axis, the normal component of stress is denoted by f_z. The shearing stress v is resolved into two components and requires two subscript letters for a complete description. The first letter indicates the direction of the normal to the plane under consideration. The second letter indicates the direction of the component of the stress. For the sides perpendicular to the z axis, the shear component in the x direction is labeled v_{zx} and that in the y direction v_{zy}.

5.3.2 Stress and Strain Components

If, for the small cube in Fig. 5.7, moments of the forces acting on it are taken a bout the x axis, considering the cube's dimensions as dx, dy, and dz, the equation of equilibrium requires that

$$v_{zy}\,dx\,dy\,dz = v_{yz}\,dx\,dy\,dz$$

(Forces are taken equal to the product of the area of the face and the stress at the center.) Two similar equations can be written for moments taken about the y axis and z axis. These equations show that

$$v_{xy} = v_{yx} \quad v_{zx} = v_{xz} \quad \text{and} \quad v_{zy} = v_{yx} \tag{5.34}$$

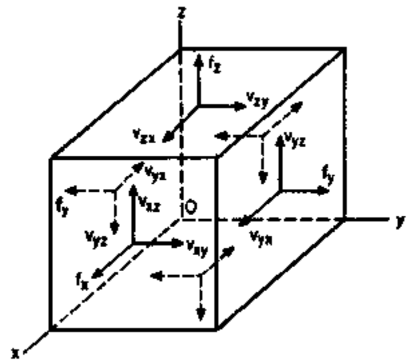


FIGURE 5.7 Normal and shear stresses in an orthogonal coordinate system.

In words, the components of shearing stress on two perpendicular faces and acting normal to the intersection of the faces are equal.

Consequently, to describe the stresses acting on the coordinate planes through a point, only six quantities need be known. These stress components are f_x, f_y, f_z $v_{xy} = v_{yx}$, $v_{yz} = v_{zy}$, and $v_{zx} = v_{xz}$.

If the cube in Fig. 5.7 is acted on only by normal stresses f_x, f_y, and f_z, from Hooke's law and the application of Poisson's ratio, the unit strains in the x, y, and z directions, in accordance with Arts. 5.2.3 and 5.2.4, are, respectively,

$$\begin{aligned}
\epsilon_x &= \frac{1}{E}[f_x - \mu(f_y + f_z)] \\
\epsilon_y &= \frac{1}{E}[f_y - \mu(f_x + f_z)] \\
\epsilon_z &= \frac{1}{E}[f_z - \mu(f_x + f_y)]
\end{aligned} \tag{5.35}$$

where μ = Poisson's ratio. If only shearing stresses act on the cube in Fig. 5.7, the distortion of the angle between edges parallel to any two coordinate axes depends only on shearing-stress components parallel to those axes. Thus, the unit shearing strains are (see Art. 5.2.4)

$$\gamma_{xy} = \frac{1}{G} v_{xy} \quad \gamma_{yz} = \frac{1}{G} v_{yx} \quad \text{and} \quad \gamma_{zx} = \frac{1}{G} v_{zx} \tag{5.36}$$

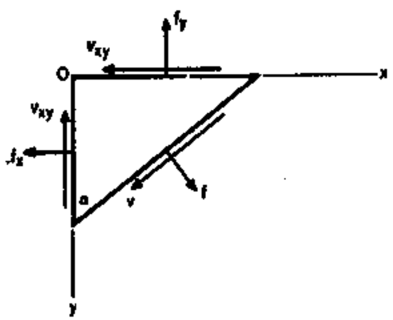


FIGURE 5.8 Normal and shear stresses at a point on a plane inclined to the axes.

5.3.3 Two-Dimensional Stress

When the six components of stress necessary to describe the stresses at a point are known (Art. 5.3.2), the stress on any inclined plane through the same point can be determined. For the case of two-dimensional stress, only three stress components need be known.

Assume, for example, that at a point O in a stressed plate, the components f_x, f_y, and v_{xy} are known (Fig. 5.8). To find the stresses for any plane through the z axis, take a plane parallel to it close to

O. This plane and the coordinate planes from a triangular prism. Then, if α is the angle the normal to the plane makes with the x axis, the normal and shearing stresses on the inclined plane, obtained by application of the equations of equilibrium, are

$$f = f_x \cos^2 \alpha + f_y \sin^2 \alpha + 2v_{xy} \sin \alpha \cos \alpha \tag{5.37}$$

$$v = v_{xy}(\cos^2 \alpha - \sin^2 \alpha) + (f_y - f_x) \sin \alpha \cos \alpha \tag{5.38}$$

Note. All structural members are three-dimensional. While two-dimensional-stress calculations may be sufficiently accurate for most practical purposes, this is not always the case. For example, although loads may create normal stresses on two perpendicular planes, a third normal stress also exists, as computed with Poisson's ratio. [See Eq. (5.35).]

5.3.4 Principal Stresses

A plane through a point on which stresses act may be assigned a direction for which the normal stress is a maximum or a minimum. There are two such positions, perpendicular to each other. And on those planes, there are no shearing stresses.

The direction in which the normal stresses become maximum or minimum are called principal directions and the corresponding normal stresses principal stresses.

To find the principal directions, set the value of v given by Eq. (5.38) equal to zero. The resulting equation is

$$\tan 2\alpha = \frac{2v_{xy}}{f_x - f_y} \tag{5.39}$$

If the x and y axes are taken in the principal directions, v_{xy} is zero. Consequently, Eqs. (5.37) and (5.38) may be simplified to

$$f = f_x \cos^2 \alpha + f_y \sin^2 \alpha \tag{5.40}$$

$$v = \tfrac{1}{2} \sin 2\alpha(f_y - f_x) \tag{5.41}$$

where f and v are, respectively, the normal and sharing stress on a plane at an angle α with the principal planes and f_x and f_y are the principal stresses.

Pure Shear. If on any two perpendicular planes only shearing stresses act, the state of stress at the point is called pure shear or simple shear. Under such conditions, the principal directions bisect the angles between the planes on which these shearing stresses occur. The principal stresses are equal in magnitude to the unit shearing stresses.

5.3.5 Maximum Shearing Stress

The maximum unit shearing stress occurs on each of two planes that bisect the angles between the planes on which the principal stresses act. The maximum share is equal to one-half the algebraic difference of the principal stresses:

$$\max v = \frac{f_1 - f_2}{2} \tag{5.42}$$

where f_1 is the maximum principal stress and f_2 the minimum.

5.3.6 Mohr's Circle

The relationship between stresses at a point may be represented conveniently on Mohr's circle (Fig. 5.9). In this diagram, normal stress f and shear stress v are taken as coordinates. Then, for each plane through the point, there will correspond a point on the circle, whose coordinates are the values of f and v for the plane.

To construct the circle given the principal stresses, mark off the principal stresses f_1 and f_2 on the f axis (points A and B in Fig. 5.9). Tensile stresses are measured to the right of the v axis and compressive stresses to the left. Construct a circle with its center on the f axis and passing through the two points representing the principal stresses. This is the Mohr's circle for the given stresses at the point under consideration.

Suppose now, we wish to find the stresses on a plane at an angle α to the plane of f_1. If a radius is drawn making an angle 2α with the f axis, the coordinates of its intersection with the circle represent the normal and sharing stresses acting on the plane.

Mohr's circle an also be plotted when the principal stresses are not known but the stresses f_x, f_y, and v_{xy}, on any two perpendicular planes, are. The procedure is to plot the two points representing these known stresses with respect to the f and v axies (points C and D in Fig. 5.10). The line joining these points is a diameter

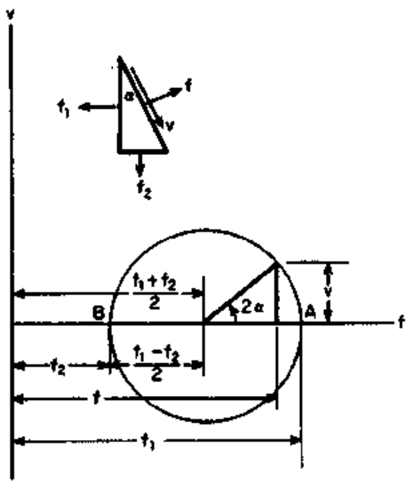


FIGURE 5.9 Mohr's circle for stresses at a point—constructed from known principal stresses.

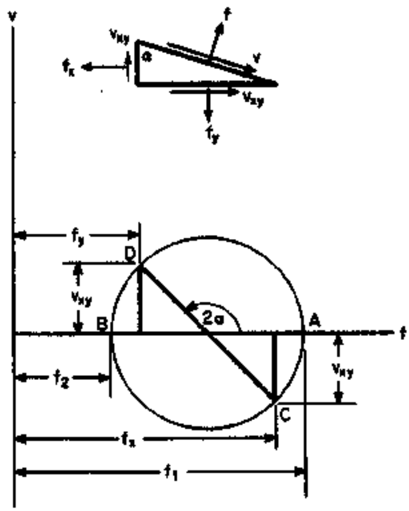


FIGURE 5.10 Stress circle constructed from two known positive stresses f_x and f_y and a shear stress v_{xy}.

of Mohr's circle. Constructing the circle on this diameter, we find the principal stresses at the intersection with the f axis (points A and B in Fig. 5.10).

For more details on the relationship of stresses and strains at a point, see Timoshenko and Goodier, "Theory of Elasticity," McGraw-Hill Publishing Company, New York.

5.4 TORSION

Forces that cause a member to twist about a longitudinal axis are called torsional loads. Simple torsion is produced only by a couple, or moment, in a plane perpendicular to the axis.

If a couple lies in a nonperpendicular plane, it can be resolved into a torsional moment, in a plane perpendicular to the axis, and bending moments, in planes through the axis.

5.4.1 Shear Center

The point in each normal section of a member through which the axis passes and about which the section twists is called the share center. The location of the shear center depends on the shape and dimensions of the cross section. If the loads on a beam do not pass through the shear center, they cause the beam to twist. See also Art. 5.5.19.

If a beam has an axis of symmetry, the shear center lies on it. In doubly symmetrical beams, the share center lies at the intersection of the two axes of symmetry and hence coincides with the centroid.

For any section composed of two narrow rectangles, such as a T beam or an angle, the shear center may be taken as the intersection of the longitudinal center lines of the rectangles.

For a channel section with one axis of symmetry, the shear center is outside the section at a distance from the centroid equal to $e(1 + h^2A/4I)$, where e is the distance from the centroid to the center of the web, h is the depth of the channel, A the cross-sectional area, and I the moment of inertia about the axis of symmetry. (The web lies between the shear center and the centroid.)

Locations of shear centers for several other sections are given in Friedrich Bleich, "Buckling Strength of Metal Structures," Chap. III, McGraw-Hill Publishing Company, New York.

5.4.2 Stresses Due to Torsion

Simple torsion is resisted by internal shearing stresses. These can be resolved into radial and tangential shearing stresses, which being normal to each other also are equal (see Art. 5.3.2). Furthermore, on planes that bisect the angles between the planes on which the shearing stresses act, there also occur compressive and tensile stresses. The magnitude of these normal stresses is equal to that of the shear. Therefore, when torsional loading is combined with other types of loading, the maximum stresses occur on inclined planes and can be computed by the methods of Arts. 5.3.3 and 5.3.6.

Circular Sections. If a circular shaft (hollow or solid) is twisted, a section that is plane before twisting remains plane after twisting. Within the proportional limit, the shearing unit stress at any point in a transverse section varies with the distance from the center of the section. The maximum shear, psi, occurs at the circumference and is given by

$$v = \frac{Tr}{J} \tag{5.43}$$

where T = torsional moment, in-lb
r = radius of section, in
J = polar moment of inertia, in^4

Polar moment of inertia of a cross section is defined by

$$J = \int \rho^2 \, dA \tag{5.44}$$

where ρ = radius from shear center to any point in the section
dA = differential area at the point

In general, J equals the sum of the moments of inertia above any two perpendicular axes through the shear center. For a solid circular section, $J = \pi r^4/2$. For a hollow circular section with diameters D and d, $J = \pi(D^4 - d^4)/32$.

Within the proportional limits, the angular twist between two points L inches apart along the axis of a circular bar is, in radians (1 rad = 57.3°):

$$\theta = \frac{TL}{GJ} \tag{5.45}$$

where G is the shearing modulus of elasticity (see Art. 5.2.4).

Noncircular Sections. If a shaft is not circular, a plane transverse section before twisting does not remain plane after twisting. The resulting warping increases the shearing stresses in some parts of the section and decreases them in others, compared wit the sharing stresses that would occur if the section remained plane. Consequently, shearing stresses in a noncircular section are not proportional to distances from the share center. In elliptical and rectangular sections, for example, maximum shear occurs on the circumference at a point nearest the shear center.

For a solid rectangular section, this maximum may be expressed in the following form:

$$v = \frac{T}{kb^2d} \tag{5.46}$$

where b = short side of rectangle, in
d = long side, in
k = constant depending on ratio of these sides;

d/b = 1.0	1.5	2.0	3	4	5	10	∞	
k = 0.208	0.231	0.246	0.258	0.267	0.282	0.291	0.312	0.333

(S. Timoshenko and J. N. Goodier, "Theory of Elasticity," McGraw-Hill Publishing Company, New York.)

Hollow Tubes. If a thin-shell hollow tube is twisted, the shearing force per unit of length on a cross section (**shear flow**) is given approximately by

$$H = \frac{T}{2A} \tag{5.47}$$

where A is the area enclosed by the mean perimeter of the tube, in^2, and the unit shearing stress is given approximately by

$$v = \frac{H}{t} = \frac{T}{2At} \tag{5.48}$$

where t is the thickness of the tube, in. For a rectangular tube with sides of unequal thickness, the total shear flow can be computed from Eq. (5.47) and the shearing stress along each side from Eq. (5.48), except at the corners, where there may be appreciable stress concentration.

Channels and I Beams. For a narrow rectangular section, the maximum shear is very nearly equal to

$$v = \frac{t^{1}\!/_{3}}{b^2 d} \tag{5.49}$$

This formula also can be used to find the maximum shearing stress due to torsion in members, such as I beams and channels, made up of thin rectangular components. Let $J = \frac{1}{3}\Sigma b^3 d$, where b is the thickness of each rectangular component and d the corresponding length. Then, the maximum shear is given approximately by

$$v = \frac{Tb'}{J} \tag{5.50}$$

where b' is the thickness of the web or the flange of the member. Maximum shear will occur at the center of one of the long sides of the rectangular part that has the greatest thickness. (A. P. Boresi, O. Sidebottom, F. B. Seely, and J. O. Smith, "Advanced Mechanics of Materials," 3d ed., John Wiley & Sons, Inc., New York.)

5.5 STRAIGHT BEAMS

Beams are the horizontal members used to support vertically applied loads across an opening. In a more general sense, they are structural members that external loads tend to bend, or curve. Usually, the term *beam* is applied to members with top continuously connected to bottom throughout their length, and those with top and bottom connected at intervals are called *trusses.* See also Structural System, Art. 1.7.

5.5.1 Types of Beams

There are many ways in which beams may be supported. Some of the more common methods are shown in Figs. 5.11 to 5.16.

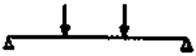


FIGURE 5.11 Simple beam.

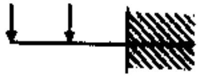


FIGURE 5.12 Cantilever beam.

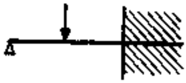


FIGURE 5.13 Beam with one end fixed.

FIGURE 5.14 Fixed-end beam.

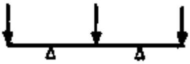


FIGURE 5.15 Beam with overhangs.

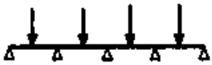


FIGURE 5.16 Continuous beam.

The beam in Fig. 5.11 is called a simply supported, or **simple beam.** It has supports near its ends, which restrain it only against vertical movement. The ends of the beam are free to rotate. When the loads have a horizontal component, or when change in length of the beam due to temperature may be important, the supports may also have to prevent horizontal motion. In that case, horizontal restraint at one support is generally sufficient.

The distance between the supports is called the **span.** The load carried by each support is called a **reaction.**

The beam in Fig. 5.12 is a **cantilever.** It has only one support, which restrains it from rotating or moving horizontally or vertically at that end. Such a support is called a **fixed end.**

If a simple support is placed under the free end of the cantilever, the **propped beam** in Fig. 5.13 results. It has one end fixed, one end simply supported.

The beam in Fig. 5.14 has both ends fixed. No rotation or vertical movement can occur at either end. In actual practice, a fully fixed end can seldom be obtained. Some rotation of the beam ends generally is permitted. Most support conditions are intermediate between those for a simple beam and those for a fixed-end beam.

In Fig. 5.15 is shown a beam that overhangs both is simple supports. The overhangs have a free end, like cantilever, but the supports permit rotation.

When a beam extends over several supports, it is called a **continuous beam** (Fig. 5.16).

Reactions for the beams in Figs. 5.11, 5.12, and 5.15 may be found from the equations of equilibrium. They are classified as **statically determinate beams** for that reason.

The equations of equilibrium, however, are not sufficient to determine the reactions of the beams in Figs. 5.13, 5.14, and 5.16. For those beams, there are more unknowns than equations. Additional equations must be obtained on the basis of deformations permitted; on the knowledge, for example, that a fixed end permits no rotation. Such beams are classified as **statically indeterminate.** Methods for finding the stresses in that type of beam are given in Arts. 5.10.4, 5.10.5, 5.11, and 5.13.

5.5.2 Reactions

As an example of the application of the equations of equilibrium (Art. 5.2.1) to the determination of the reactions of a statically determinate beam, we shall compute the reactions of the 60-ft-long beam with overhangs in Fig. 5.17. This beam carries a uniform load of 200 lb/lin ft over its entire length and several concentrated loads. The supports are 36 ft apart.

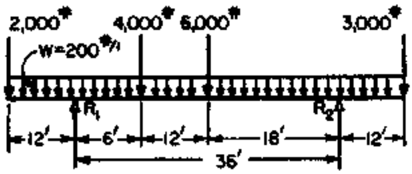


FIGURE 5.17 Beam with overhangs loaded with both uniform and concentrated loads.

To find reaction R_1, we take moments about R_2 and equate the sum of the moments to zero (clockwise rotation is considered positive, counterclockwise, negative):

$$-2000 \times 48 + 36R_1 - 4000 \times 30 - 6000 \times 18 + 3000 \times 12$$
$$-200 \times 60 \times 18 = 0$$
$$R_1 = 14{,}000 \text{ lb}$$

In this calculation, the moment of the uniform load was found by taking the moment of its resultant, which acts at the center of the beam.

To find R_2, we can either take moments about R_1 or use the equation $\Sigma V = 0$. It is generally preferable to apply the moment equation and use the other equation as a check.

$$3000 \times 48 - 36R_2 + 6000 \times 18 + 4000 \times 6 - 2000 \times 12$$
$$+ 200 \times 60 \times 18 = 0$$
$$R_2 = 13{,}000 \text{ lb}$$

As a check, we note that the sum of the reactions must equal the total applied load:

$$14{,}000 + 13{,}000 = 2000 + 4000 + 6000 + 3000 + 12{,}000$$
$$27{,}000 = 27{,}000$$

5.5.3 Internal Forces

Since a beam is in equilibrium under the forces applied to it, it is evident that at every section internal forces are acting to prevent motion. For example, suppose we cut the beam in Fig. 5.17 vertically just to the right of its center. If we total the external forces, including the reaction, to the left of this cut (see Fig. 5.18*a*), we find there is an unbalanced downward load of 4000 lb. Evidently, at the cut section, an upward-acting internal force of 4000 lb must be present to maintain equilibrium. Again, if we take moments of the external forces about the section, we find an unbalanced moment of 54,000 ft-lb. So there must be an internal moment of 54,000 ft-lb acting to maintain equilibrium.

This internal, or resisting, moment is produced by a couple consisting of a force C acting on the top part of the beam and an equal but opposite force T acting on

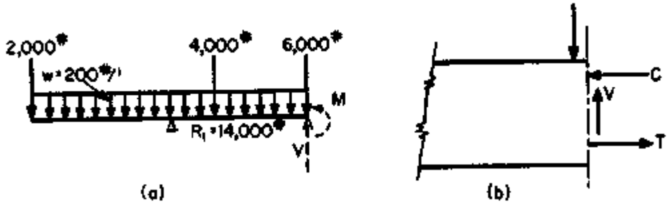


FIGURE 5.18 Portions of a beam are held in equilibrium by internal stresses.

the bottom part (Fig. 18*b*). The top force is the resultant of compressive stresses acting over the upper portion of the beam, and the bottom force is the resultant of tensile stresses acting over the bottom part. The surface at which the stresses change from compression to tension—where the stress is zero—is called the **neutral surface.**

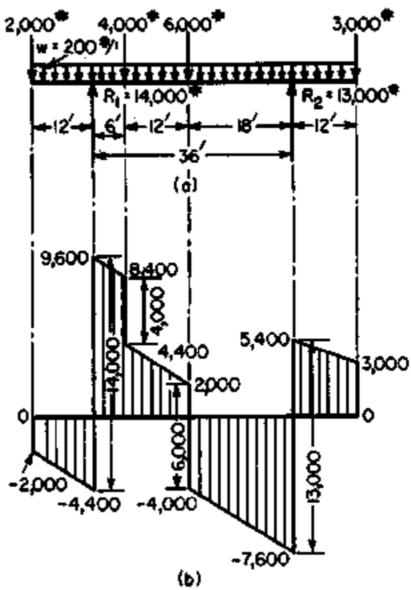


FIGURE 5.19 .Shear diagram for the beam with loads shown in Fig. 5.17.

5.5.4 Shear Diagrams

The unbalanced external vertical force at a section is called the shear. It is equal to the algebraic sum of the forces that lie on either side of the section. Upward acting forces on the left of the section are considered positive, downward forces negative; signs are reversed for forces on the right.

A diagram in which the shear at every point along the length of a beam is plotted as an ordinate is called a shear diagram. The shear diagram for the beam in Fig. 5.17 is shown in Fig. 5.19*b*.

The diagram was plotted starting from the left end. The 2000-lb load was plotted downward to a convenient scale. Then, the shear at the next concentrated load—the left support—was determined. This equals $-2000 - 200 \times 12$, or -4400 lb. In passing from must to the left of the support to a point just to the right, however, the shear changes by the magnitude of the reaction. Hence, on the right-hand side of the left support the shear is $-4400 + 14{,}000$, or 9600 lb. At the next concentrated load, the shear is $9600 - 200 \times 6$, or 8400 lb. In passing the 4000-lb load, however, the shear changes to $8400 - 4000$, or 4400 lb. Proceeding in this manner to the right end of the beam, we terminate with a shear of 3000 lb, equal to the load on the free end there.

It should be noted that the shear diagram for a uniform load is a straight line sloping downward to the right (see Fig. 5.21). Therefore, the shear diagram was completed by connecting the plotted points with straight lines.

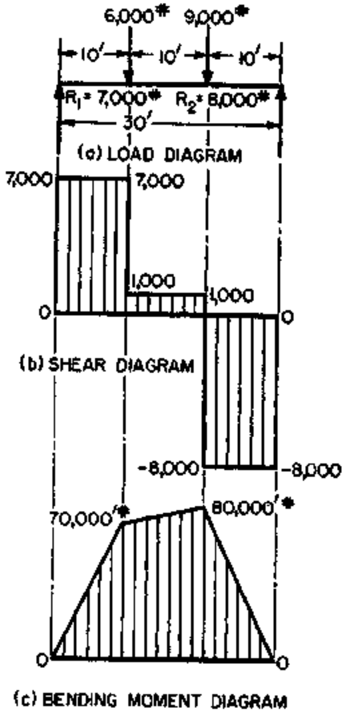


FIGURE 5.20 Shear and moment diagrams for a simply supported beam with concentrated loads.

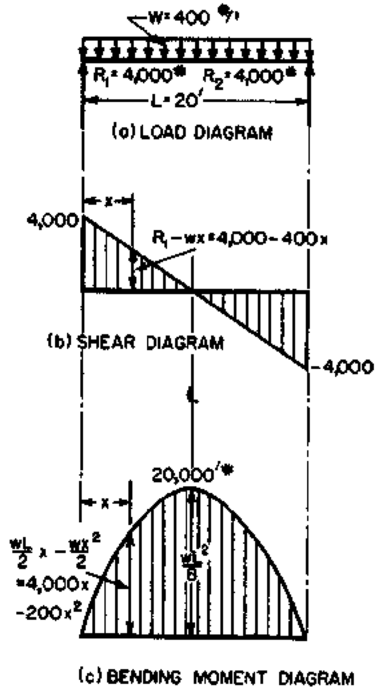


FIGURE 5.21 Shear and moment diagrams for a simply supported, uniformly loaded beam.

Shear diagrams for commonly encountered loading conditions are given in Figs. 5.30 to 5.41.

5.5.5 Bending-Moment Diagrams

The unbalanced moment of the external forces about a vertical section through a beam is called the bending moment. It is equal to the algebraic sum of the moments about the section of the external forces that lie on one side of the section. Clockwise moments are considered positive, counterclockwise moments negative, when the forces considered lie on the left of the section. Thus, when the bending moment is positive, the bottom of the beam is in tension.

A diagram in which the bending moment at every point along the length of a beam is plotted as an ordinate is called a bending-moment diagram.

Figure 5.20*c* is the bending-moment diagram for the beam loaded with concentrated loads only in Fig. 5.20*a*. The bending moment at the supports for this simply supported beam obviously is zero. Between the supports and the first load, the bending moment is proportional to the distance from the support, since it is equal to the reaction times the distance from the support. Hence the bending-moment diagram for this portion of the beam is a sloping straight line.

The bending moment under the 6000-lb load in Fig. 5.20*a* considering only the force to the left is 7000 × 10, or 70,000 ft-lb. The bending-moment diagram, then, between the left support and the first concentrated load is a straight line rising from zero at the left end of the beam to 70,000 ft-lb, plotted to a convenient scale, under the 6000-lb load.

The bending moment under the 9000-lb load, considering the forces on the left of it, is 7000 × 20 − 6000 × 10, or 80,000 ft-lb. (It could have been more easily obtained by considering only the force on the right, reversing the sign convention: 8000 × 10 = 80,000 ft-lb.) Since there are no loads between the two concentrated loads, the bending-moment diagram between the two sections is a sloping straight line.

If the bending moment and shear are known at any section of a beam, the bending moment at any other section may be computed, providing there are no unknown forces between the two sections. The rule is:

The bending moment at any section of a beam is equal to the bending moment at any section to the left, plus the shear at that section times the distance between sections, minus the moments of intervening loads. If the section with known moment and share is on the right, the sign convention must be reversed.

For example, the bending moment under the 9000-lb load in Fig. 5.20*a* could also have been obtained from the moment under the 6000-lb load and the shear to the right of the 6000-lb load given in the shear diagram (Fig. 5.20*b*). Thus, 80,000 = 70,000 + 1000 × 10. If there had been any other loads between the two concentrated loads, the moment of these loads about the section under the 9000-lb load would have been subtracted.

Bending-moment diagrams for commonly encountered loading conditions are given in Figs. 5.30 to 5.41. These may be combined to obtain bending moments for other loads.

5.5.6 Moments in Uniformly Loaded Beams

When a bean carries a uniform load, the bending-moment diagram does not consist of straight lines. Consider, for example, the beam in Fig. 5.21*a*, which carries a uniform load over its entire length. As shown in Fig. 5.21*c*, the bending-moment diagram for this beam is a parabola.

The reactions at both ends of a simply supported, uniformly loaded beam are both equal to $wL/2 = W/2$, where w is the uniform load in pounds per linear foot, $W = wL$ is the total load on the beam, and L is the span.

The shear at any distance x from the left support is $R_1\ wx = wL/2 - wx$ (see Fig. 5.21*b*). Equating this expression to zero, we find that there is no shear at the center of the beam.

The bending moment at any distance x from the left support is

$$M = R_1 x - wx\left(\frac{x}{2}\right) = \frac{wLx}{2} - \frac{wx^2}{2} = \frac{w}{2}\,x(L - x) \qquad (5.51)$$

Hence:

The bending moment at any section of a simply supported, uniformly loaded beam is equal to one-half the product of the load per linear foot and the distances to the section from both supports.

The maximum value of the bending moment occurs at the center of the beam. It is equal to $wL^2/8 = WL/8$.

5.5.7 Shear-Moment Relationship

The slope of the bending-moment curve for any point on a beam is equal to the shear at that point; i.e.,

$$V = \frac{dM}{dx} \tag{5.52}$$

Since maximum bending moment occurs when the slope changes sign, or passes through zero, maximum moment (positive or negative) occurs at the point of zero shear.

After integration, Eq. (5.52) may also be written

$$M_1 - M_2 = \int_{x2}^{x1} V\,dx \tag{5.53}$$

5.5.8 Moving Loads and Influence Lines

One of the most helpful devices for solving problems involving variable or moving loads is an influence line. Whereas shear and moment diagrams evaluate the effect of loads at all sections of a structure, an influence line indicates the effect at a given section of a unit load placed at any point on the structure.

For example, to plot the influence line for bending moment at some point *A* on a beam, a unit load is applied at some point *B*. The bending moment is *A* due to the unit load at *B* is plotted as an ordinate to a convenient scale at *B*. The same procedure is followed at every point along the beam and a curve is drawn through the points thus obtained.

Actually, the unit load need not be placed at every point. The equation of the influence line can be determined by placing the load at an arbitrary point and computing the bending moment in general terms. (See also Art. 5.10.5.)

Suppose we wish to draw the influence line for reaction at *A* for a simple beam *AB* (Fig. 5.22*a*). We place a unit load at an arbitrary distance of xL from *B*. The reaction at *A* due to this load is $1\ xL/L = x$. Then, $R_A = x$ is the equation of the influence line. It represents a straight line sloping upward from zero at *B* to unity at *A* (Fig. 5.22*a*). In other words, as the unit load moves across the beam, the reaction at *A* increases from zero to unity in proportion to the distance of the load from *B*.

Figure 5.22*b* shows the influence line for bending moment at the center of a beam. It resembles in appearance the bending-moment diagram for a load at the center of the beam, but its significance is entirely different. Each ordinate gives the moment at midspan for a load at the corresponding location. It indicates that, if a unit load is placed at a distance xL from one end, it produces a bending moment of $\frac{1}{2}\,xL$ at the center of the span.

Figure 5.22*c* shows the influence line for shear at the quarter point of a beam. When the load is to the right of the quarter point, the shear is positive and equal to the left reaction. When the load is to the left, the shear is negative and equal to the right reaction.

The diagram indicates that, to produce maximum shear at the quarter point, loads should be placed only to the right of the quarter point, with the largest load at the quarter point, if possible. For a uniform load, maximum shear results when the load extends from the right end of the beam to the quarter point.

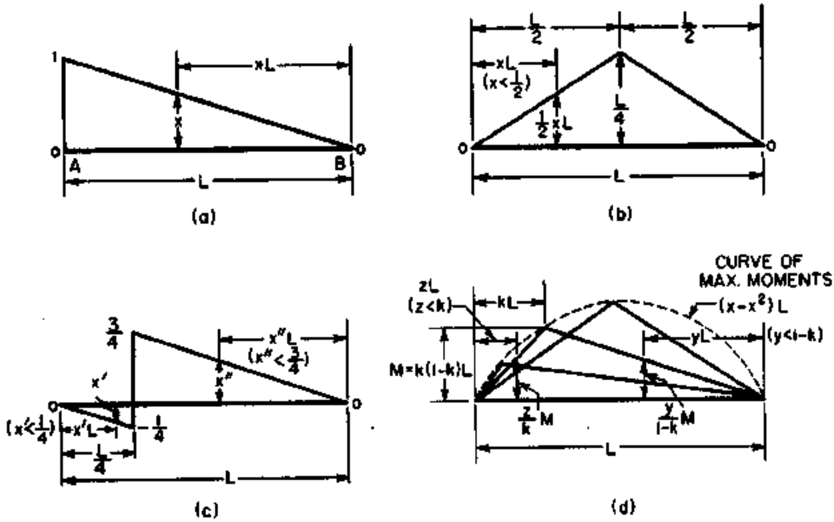


FIGURE 5.22 Influence lines for simple beam AB for (a) reaction at A; (b) midspan bending moment; (c) quarter-point shear; and (d) bending moments for unit load at several points on the beam.

Suppose, for example, that the beam is a crane girder with a span of 60 ft. The wheel loads are 20 and 10 kips, respectively, and are spaced 5 ft apart. For maximum shear at the quarter point, the wheels should be placed with the 20-kip wheel at that point and the 10-kip wheel to the right of it. The corresponding ordinates of the influence line (Fig. 5.22c) are $3/4$ and $40/45 \times 3/4$. Hence, the maximum shear is $20 \times 3/4 + 10 \times 40/45 \times 3/4 = 21.7$ kips.

Figure 5.22d shows influence lines for bending moment at several points on a beam. It is noteworthy that the apexes of the diagrams fall on a parabola, as shown by the dashed line. This indicates that the maximum moment produced at any given section by a single concentrated load moving across a beam occurs when the load is at that section. The magnitude of the maximum moment increases when the section is moved toward midspan, in accordance with the equation shown in Fig. 5.22d for the parabola.

5.5.9 Maximum Bending Moment

When there is more than one load on the span, the influence line is useful in developing a criterion for determining the position of the loads for which the bending moment is a maximum at a given section.

Maximum bending moment will occur at a section C of a simple beam as loads move across it when one of the loads is at C. The proper load to place at C is the one for which the expression $W_a/a - W_b/b$ (Fig. 5.23) changes sign as that load passes from one side of C to the other.

When several loads move across a simple beam, the maximum bending moment produced in the beam may be near but not necessarily at midspan. To find the maximum moment, first determine the position of the loads for maximum moment

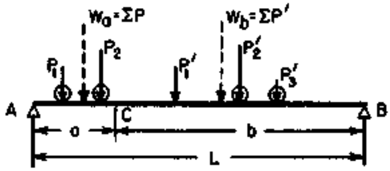


FIGURE 5.23 .Moving loads on simple beam *AB* ae placed for maximum bending moment at point *C* on the beam.

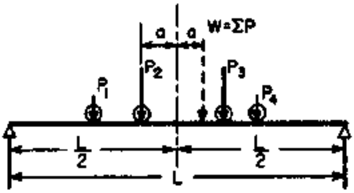


FIGURE 5.24 Moving loads are placed to subject a simple beam to the largest possible bending moment.

at midspan. Then shift the loads until the load P_2 that was at the center of the beam is as far from midspan as the resultant of all the loads on the span is on the other side of midspan (Fig. 5.24). Maximum moment will occur under P_2.

When other loads move on or off the span during the shift of P_2 away from midspan, it may be necessary to investigate the moment under one of the other loads when it and the resultant are equidistant from midspan.

5.5.10 Bending Stresses in a Beam

To derive the commonly used flexure formula for computing the bending stresses in a beam, we have to make the following assumptions:

1. The unit stress at a point in any plane parallel to the neutral surface of a beam is proportional to the unit strain in the plane at the point.
2. The modulus of elasticity in tension is the same as that in compression.
3. The total and unit axial strain in any plane parallel to the neutral surface are both proportional to the distance of that plane from the neutral surface. (Cross sections that are plane before bending remain plane after bending. This requires that all planes have the same length before bending; thus, that the beam be straight.)
4. The loads act in a plane containing the centroidal axis of the beam and are perpendicular to that axis. Furthermore, the neutral surface is perpendicular to the plane of the loads. Thus, the plane of the loads must contain an axis of symmetry of each cross section of the beam. (The flexure formula does not apply to a beam loaded unsymmetrically. See Arts. 5.5.18 and 5.5.19.)
5. The beam is proportioned to preclude prior failure or serious deformation by torsion, local buckling, shear, or any cause other than bending.

Equating the bending moment to the resisting moment due to the internal stresses at any section of a beam yields

$$M = \frac{fI}{C} \tag{5.54}$$

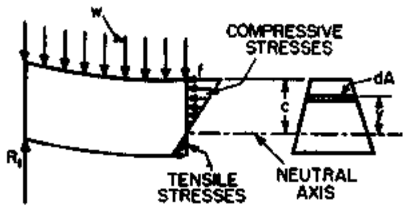


FIGURE 5.25 Unit stresses on a beam cross section caused by bending of the beam.

M is the bending moment at the section, f is the normal unit stress in a plane at a distance c from the neutral axis (Fig. 5.25), and I is the moment of inertia of the cross section with respect to the neutral axis. If f is given in pounds per square inch (psi), I in in^4, and c in inches, then M will be in inch-pounds. For maximum unit stress, c is the distance to the outermost fiber. See also Arts. 5.5.11 and 5.5.12.

5.5.11 Moment of Inertia

The neutral axis in a symmetrical beam is coincidental with the centroidal axis; i.e., at any section the neutral axis is so located that

$$\int y \, dA = 0 \tag{5.55}$$

where dA is a differential area parallel to the axis (Fig. 5.25), y is its distance from the axis, and the summation is taken over the entire cross section.

Moment of inertia with respect to the neutral axis is given by

$$I = \int y^2 \, dA \tag{5.56}$$

Values of I for several common types of cross section are given in Fig. 5.26. Values for structural-steel sections are presented in manuals of the American Institute of Steel Construction, Chicago, Ill. When the moments of inertia of other types of sections are needed, they can be computed directly by application of Eq. (5.56) or by braking the section up into components for which the moment of inertia is known.

If I is the moment of inertia about the neutral axis, A the cross-sectional area, and d the distance between that axis and a parallel axis in the plane of the cross section, then the moment of inertia about the parallel axis is

$$I' = I + Ad^2 \tag{5.57}$$

With this equation, the known moment of inertia of a component of a section about the neutral axis of the component can be transferred to the neutral axis of the complete section. Then, summing up the transferred moments of inertia for all the components yields the moment of inertia of the complete section.

When the moments of inertia of an area with respect to any two perpendicular axes are known, the moment of inertia with respect to any other axis passing through the point of intersection of the two axes may be obtained through the use

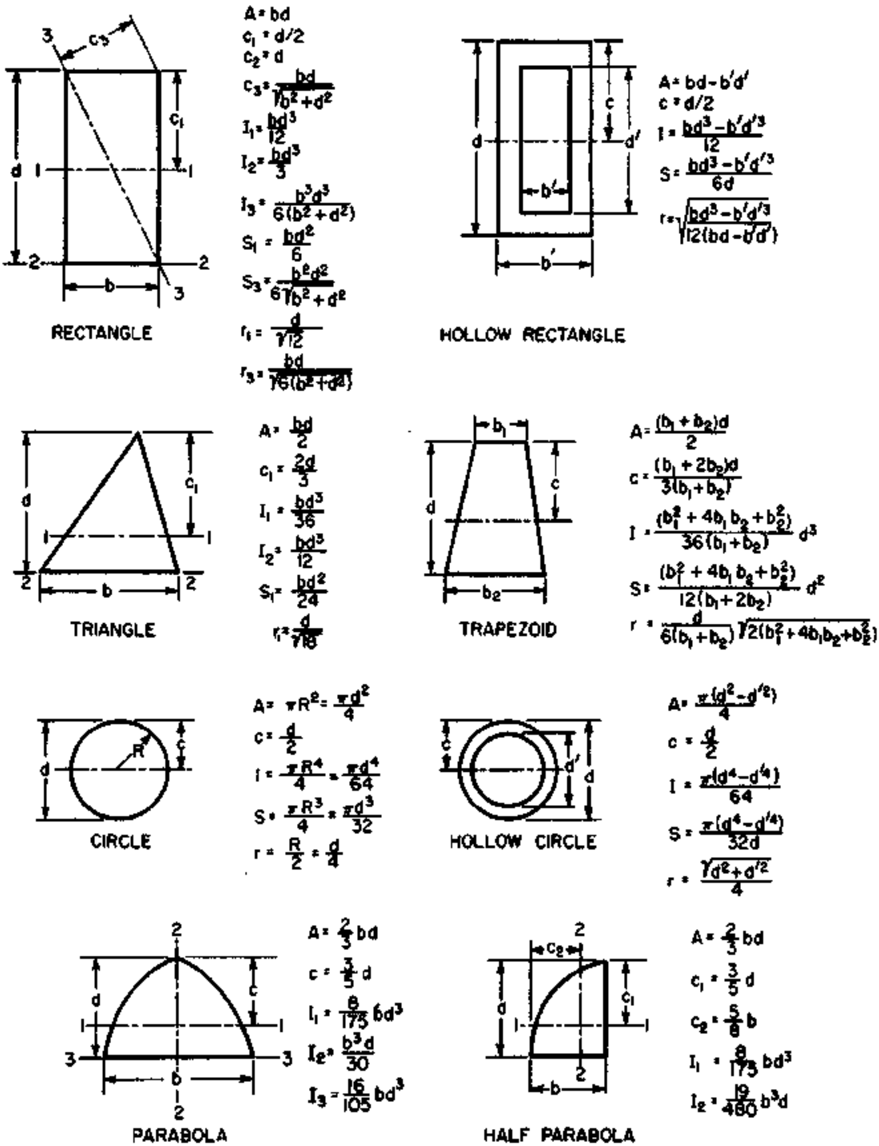


FIGURE 5.26 Geometric properties of various cross sections.

of Mohr's circle, as for stresses (Fig. 5.10). In this analog, I_x corresponds with f_x, I_y with f_y, and the **product of inertia** I_{xy} with v_{xy} (Art. 5.3.6).

$$I_{xy} = \int xy\, dA \tag{5.58}$$

The two perpendicular axes through a point about which the moments of inertia are a maximum and a minimum are called the principal axes. The products of inertia are zero for the principal axes.

5.5.12 Section Modulus

The ratio $S = I/c$ in Eq. (5.54) is called the section modulus. I is the moment of inertia of the cross section about the neutral axis and c the distance from the neutral axis to the outermost fiber. Values of S for common types of sections are given in Fig. 5.26.

5.5.13 Shearing Stresses in a Beam

The vertical shear at any section of a beam is resisted by nonuniformly distributed, vertical unit stresses (Fig. 5.27). At every point in the section, there is also a horizontal unit stress, which is equal in magnitude to the vertical unit shearing stress there [see Eq. (5.34)].

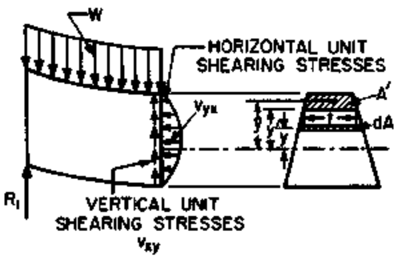


FIGURE 5.27 Unit shearing stresses on a beam cross section.

At any distances y' from the neutral axis, both the horizontal and vertical shearing unit stresses are equal to

$$v = \frac{V}{It} A'\bar{y} \tag{5.59}$$

where V = vertical shear at the cross section
t = thickness of beam at distance y' from neutral axis
I = moment of inertia about neutral axis
A' = area between the outermost fiber and the fiber for which the shearing stress is being computed
$\bar{y}$ = distance of center of gravity of this area from the neutral axis (Fig. 5.27)

For a rectangular beam with width b and depth d, the maximum shearing stress occurs at middepth. Its magnitude is

$$v = \frac{12V}{bd^3b}\frac{bd^2}{8} = \frac{3}{2}\frac{V}{bd}$$

That is, the maximum shear stress is 50% greater than the average shear stress on the section. Similarly, for a circular beam, the maximum is one-third greater than the average. For an I beam, however, the maximum shearing stress in the web is not appreciably greater than the average for the web section alone, if it is assumed that the flanges take no shear.

5.5.14 Combined Shear and Bending Stress

For deep beams on short spans and beams made of low-strength materials, it is sometimes necessary to determine the maximum stress f' on an inclined plane caused by a combination of shear and bending stress—v and f, respectively. This stress f', which may be either tension or compression, is greater than the normal stress. Its value may be obtained by application of Mohr's circle (Art. 5.3.6), as indicated in Fig. 5.10, but with $f_y = 0$, and is

$$f' = \frac{f}{2} + \sqrt{v^2 + \left(\frac{f}{2}\right)^2} \tag{5.60}$$

5.5.15 Beam Deflections

When a beam is loaded, it deflects. The new position of its longitudinal centroidal axis is called the **elastic curve.**

At any point of the elastic curve, the radius of curvature is given by

$$R = \frac{EI}{M} \tag{5.61}$$

where M = bending moment at the point
E = modulus of elasticity
I = moment of inertia of the cross section about the neutral axis

Since the slope dy/dx of the curve is small, its square may be neglected, so that, for all practical purposes, $1/R$ may be taken equal to d^2y/dx^2, where y is the deflection of a point on the curve at a distance x from the origin of coordinates. Hence, Eq. (5.61) may be rewritten

$$M = EI\,\frac{d^2y}{dx^2} \tag{5.62}$$

To obtain the slope and deflection of a beam, this equation may be integrated, with M expressed as a function of x. Constants introduced during the integration must be evaluated in terms of known points and slopes of the elastic curve.

Equation (5.62), in turn, may be rewritten after one integration as

$$\theta_B - \theta_A = \int_A^B \frac{M}{EI}\,dx \tag{5.63}$$

in which θ_A and θ_B are the slopes of the elastic curve at any two points A and B. If the slope is zero at one of the points, the integral in Eq. (5.63) gives the slope of the elastic curve at the other. It should be noted that the integral represents the area of the bending-moment diagram between A and B with each ordinate divided by EI.

The **tangential deviation** t of a point on the elastic curve is the distance of this point, measured in a direction perpendicular to the original position of the beam, from a tangent drawn at some other point on the elastic curve.

$$t_B - t_A = \int_A^B \frac{Mx}{EI}\, dx \tag{5.64}$$

Equation (5.64) indicates that the tangential deviation of any point with respect to a second point on the elastic curve equals the moment about the first point of the M/EI diagram between the two points. The **moment-area method** for determining the deflection of beams is a technique in which Eqs. (5.63) and (5.64) are utilized.

Suppose, for example, the deflection at midspan is to be computed for a beam of uniform cross section with a concentrated load at the center (Fig. 5.28).

Since the deflection at midspan for this loading is the maximum for the span, the slope of the elastic curve at the center of the beam is zero; i.e., the tangent is parallel to the undeflected position of the beam. Hence, the deviation of either support from the midspan tangent is equal to the deflection at the center of the beam. Then, by the moment-area theorem [Eq. (5.64)], the deflection y_c is given by the moment about either support of the area of the M/EI diagram included between an ordinate at the center of the beam and that support.

$$y_c = \frac{1}{2}\frac{PL}{4EI}\frac{L}{2}\frac{2}{3}\frac{L}{2} = \frac{PL^3}{48EI}$$

Suppose now, the deflection y at any point D at a distance xL from the left support (Fig. 5.28) is to be determined. Referring to the sketch, we note that the distance DE from the undeflected point of D to the tangent to the elastic curve at support A is given by

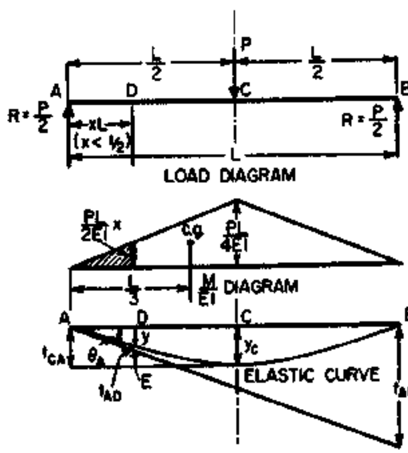


FIGURE 5.28 Load and M/EI diagrams and elastic curve for a simple beam with mispan load.

$$y + t_{AD} = xt_{AB}$$

where t_{AD} is the tangential deviation of D from the tangent at A and t_{AB} is the tangential deviation of B from that tangent. This equation, which is perfectly general for the deflection of any point of a simple beam, no matter how loaded, may be rewritten to give the deflection directly:

$$y = xt_{AB} - t_{AD} \tag{5.65}$$

But t_{AB} is the moment of the area of the M/EI diagram for the whole beam about support B. And t_{AD} is the moment about D of the area of the M/EI diagram included between ordinates at A and D. Hence

$$y = x\frac{1}{2}\frac{PL}{4EI}\frac{L}{2}\left(\frac{2}{3} + \frac{1}{3}\right)L - \frac{1}{2}\frac{PLx}{2EI}xL\frac{xL}{3} = \frac{PL^3}{48EI}x(3 - 4x^2)$$

It is also noteworthy that, since the tangential deviations are very small distances, the slope of the elastic curve at A is given by

$$\theta_A = \frac{t_{AB}}{L} \tag{5.66}$$

This holds, in general, for all simple beams regardless of the type of loading.

The procedure followed in applying Eq. (5.65) to the deflection of the loaded beam in Fig. 5.28 is equivalent to finding the bending moment at D with the M/EI diagram serving as the load diagram. The technique of applying the M/EI diagram as a load and determining the deflection as a bending moment is known as the **conjugate-beam method.**

The conjugate beam must have the same length as the given beam; it must be in equilibrium with the M/EI load and the reactions produced by the load; and the bending moment at any section must be equal to the deflection of the given beam at the corresponding section. The last requirement is equivalent to requiring that the shear at any section of the conjugate beam with the M/EI load be equal to the slope of the elastic curve at the corresponding section of the given beam. Figure 5.29 shows the conjugates for various types of beams.

Deflections for several types of loading on simple beams are given in Figs. 5.30 to 5.35 and for overhanging beams and cantilevers in Figs. 5.36 to 5.41.

When a beam carries a number of loads of different types, the most convenient method of computing its deflection generally is to find the deflections separately for the uniform and concentrated loads and add them up.

For several concentrated loads, the easiest solution is to apply the reciprocal theorem (Art. 5.10.5). According to this theorem, if a concentrated load is applied to a beam at a point A, the deflection it produces at point B is equal to the deflection at A for the same load applied at $B(d_{AB} = d_{BA})$.

Suppose, for example, the midspan deflection is to be computed. Then, assume each load in turn applied at the center of the beam and compute the deflection at the point where it originally was applied from the equation of the elastic curve given in Fig. 5.33. The sum of these deflections is the total midspan deflection.

Another method for computing deflections of beams is presented in Art. 5.10.4. This method may also be applied to determining the deflection of a beam due to shear.

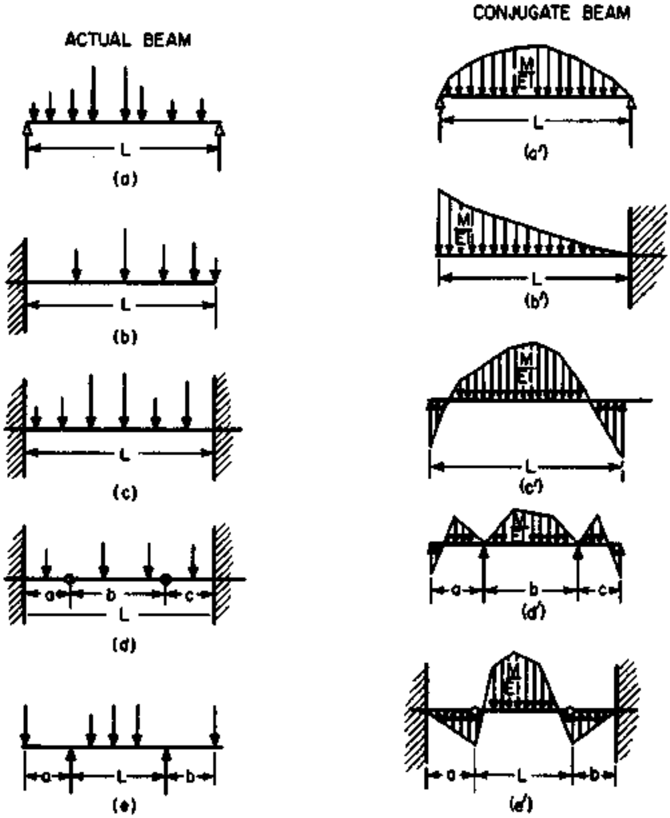


FIGURE 5.29 Various types of beams and corresponding conjugate beams.

5.5.16 Combined Axial and Bending Loads

For stiff beams, subjected to both transverse and axial loading, the stresses are given by the principle of superposition if the deflection due to bending may be neglected without serious error. That is, the total stress is given with sufficient accuracy at any section by the sum of the axial stress and the bending stresses. The maximum stress equals

$$f = \frac{P}{A} + \frac{Mc}{I} \tag{5.67}$$

where P = axial load
A = cross-sectional area
M = maximum bending moment
c = distance from neutral axis to outermost surface at the section where maximum moment occurs
I = moment of inertia of cross section about neutral axis at that section

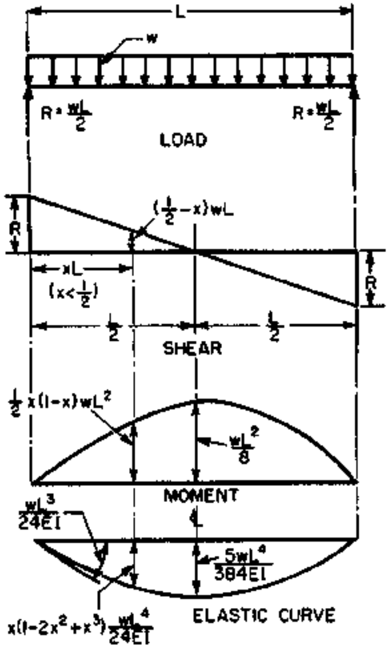


FIGURE 5.30 Uniform load over the whole span of a simple beam.

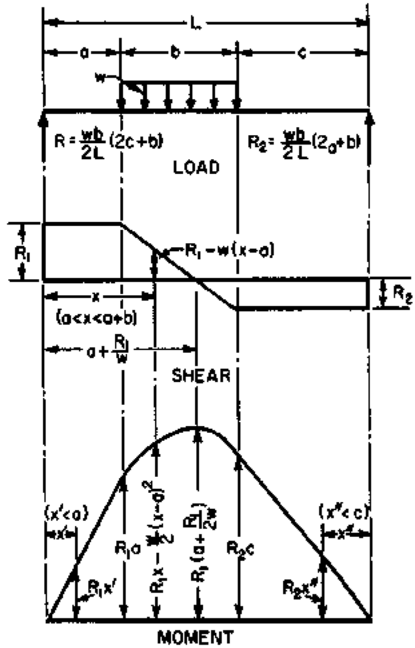


FIGURE 5.31 Uniform load over only part of a simple beam.

When the deflection due to bending is large and the axial load produces bending stresses that cannot be neglected, the maximum stress is given by

$$f = \frac{P}{A} + (M + Pd)\frac{c}{I} \tag{5.68}$$

where d is the deflection of the beam. For axial compression, the moment Pd should be given the same sign as M, and for tension, the opposite sign, but the minimum value of $M + Pd$ is zero. The deflection d for axial compression and bending can be obtained by applying Eq. (5.62). (S. Timoshenko and J. M. Gere, "Theory of Elastic Stability," McGraw-Hill Publishing company, New York; Friedrich Bleich, "Buckling Strength of Metal Structures," McGraw-Hill Publishing Company, New York.) However, it may be closely approximated by

$$d = \frac{d_o}{1 - (P/P_c)} \tag{5.69}$$

where d_o = deflection for the transverse loading alone
P_c = the critical buckling load $\pi^2 EI/L^2$ (see Art. 5.7.2)

5.5.17 Eccentric Loading

An eccentric longitudinal load in the plane of symmetry produces a bending moment Pe where e is the distance of the load from the centroidal axis. The total unit

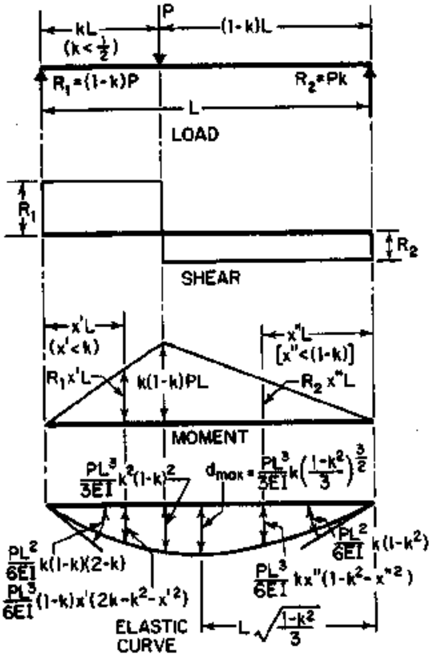


FIGURE 5.32 Concentrated load at any point of a simple beam.

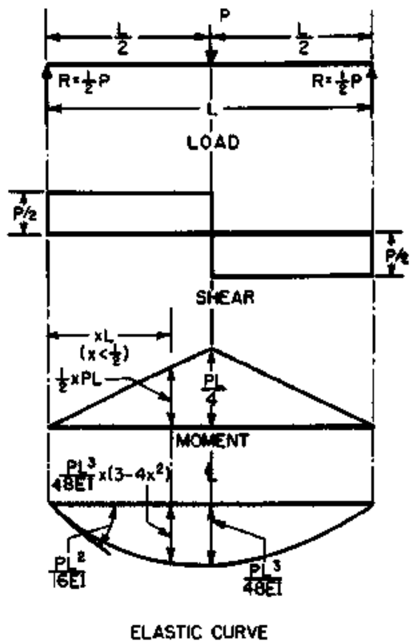


FIGURE 5.33 Concentrated load at midspan of a simple beam.

stress is the sum of the stress due to this moment and the stress due to P applied as an axial load:

$$f = \frac{P}{A} \pm \frac{Pec}{I} = \frac{P}{A}\left(1 \pm \frac{ec}{r^2}\right) \tag{5.70}$$

where A = cross-sectional area
c = distance from neutral axis to outermost fiber
I = moment of inertia of cross section about neutral axis
r = **radius of gyration,** which is equal to $\sqrt{I/A}$

Figure 5.26 gives values of the radius of gyration for some commonly used cross sections.

For an axial compression load, if there is to be no tension on the cross section, e should not exceed r^2/c. For a rectangular section with width b and depth d, the eccentricity, therefore, should be less than $b/6$ and $d/6$; i.e., the load should not be applied outside the middle third. For a circular cross section with diameter D, the eccentricity should not exceed $D/8$.

When the eccentric longitudinal load produces a deflection too large to be neglected in computing the bending stress, account must be taken of the additional bending moment Pd, where d is the deflection. This deflection may be computed by employing Eq. (5.62) or closely approximated by

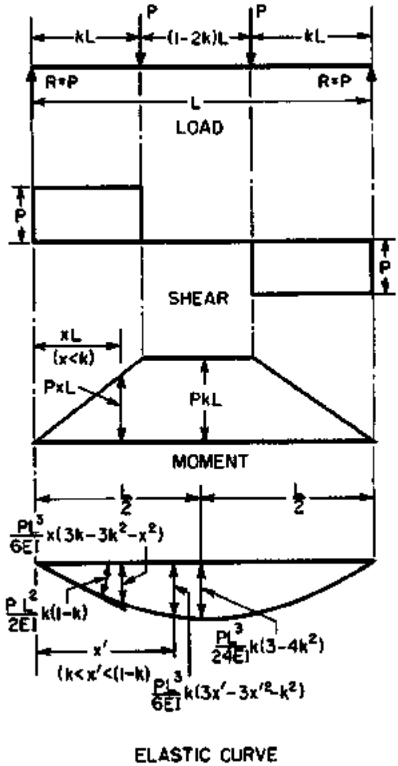


FIGURE 5.34 Two equal concentrated loads on a simple beam.

$$d = \frac{4eP/P_c}{\pi(1 - P/P_c)} \tag{5.71}$$

P_c is the critical buckling load $\pi^2 EI/L^2$ (see Art. 5.7.2).

If the load P does not lie in a plane containing an axis of symmetry, it produces bending about the two principal axes through the centroid of the cross section. The stresses are given by

$$f = \frac{P}{A} \pm \frac{Pe_x c_x}{I_y} \pm \frac{Pe_y c_y}{I_x} \tag{5.72}$$

where A = cross-sectional area
e_x = eccentricity with respect to principal axis YY
e_y = eccentricity with respect to principal axis XX
c_x = distance from YY to outermost fiber
c_y = distance from XX to outermost fiber
I_x = moment of inertia about XX
I_y = moment of inertia about YY

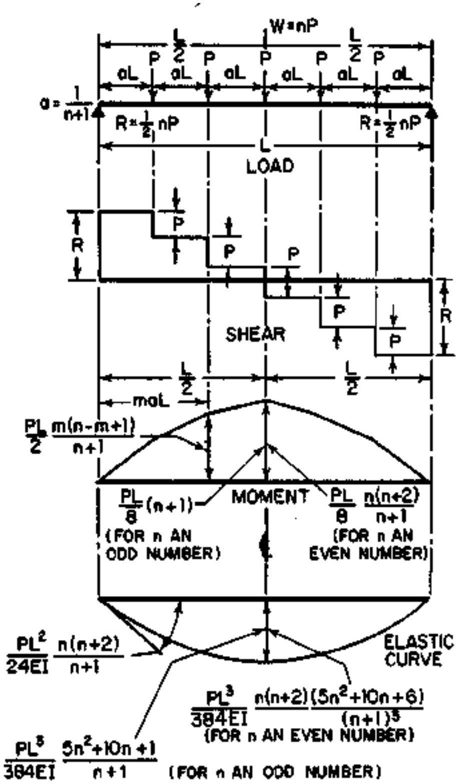


FIGURE 5.35 Several equal concentrated loads on a simple beam.

The principal axes are the two perpendicular axes through the centroid for which the moments of inertia are a maximum or a minimum and for which the products of inertia are zero.

5.5.18 Unsymmetrical Bending

Bending caused by loads that do not lie in a plane containing a principal axis of each cross section of a beam is called unsymmetrical bending. If the bending axis of the beam lies in the plane of the loads, to preclude torsion (see Art. 5.4.1), and if the loads are perpendicular to the bending axis, to preclude axial components, the stress at any point in a cross section is given by

$$f = \frac{M_x y}{I_x} \pm \frac{M_y x}{I_y} \tag{5.73}$$

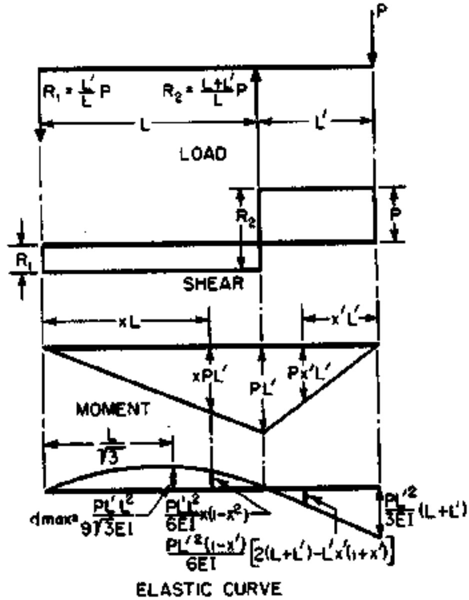


FIGURE 5.36 Concentrated load at the end of a beam overhang.

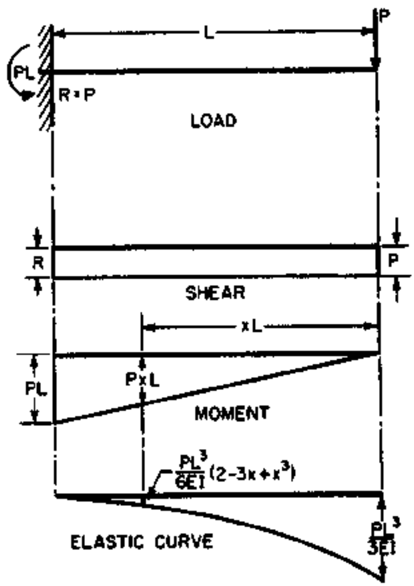


FIGURE 5.37 Concentrated load at the end of a cantilever.

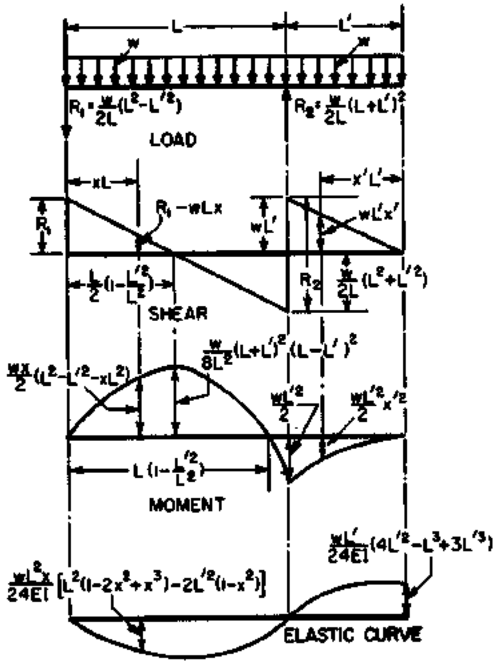


FIGURE 5.38 Uniform load over the full length of a beam with overhang.

where M_x = bending moment about principal axis *XX*
M_y = bending moment about principal axis *YY*
x = distance from point for which stress is to be computed to *YY* axis
y = distance from point to *XX* axis
I_x = moment of inertia of the cross section about *XX*
I_y = moment of inertia about *YY*

If the plane of the loads makes an angle θ with a principal plane, the neutral surface will form an angle α with the other principal plane such that

$$\tan \alpha = \frac{I_x}{I_y} \tan \theta \qquad (5.74)$$

5.5.19 Beams with Unsymmetrical Sections

In the derivation of the flexure formula $f = Mc/I$ [Eq. (5.54)], the assumption is made that the beam bends, without twisting, in the plane of the loads and that the neutral surface is perpendicular to the plane of the loads. These assumptions are correct for beams with cross sections symmetrical about two axes when the plane of the loads contains one of these axes. They are not necessarily true for beams that are not doubly symmetrical. The reason is that in beams that are doubly sym-

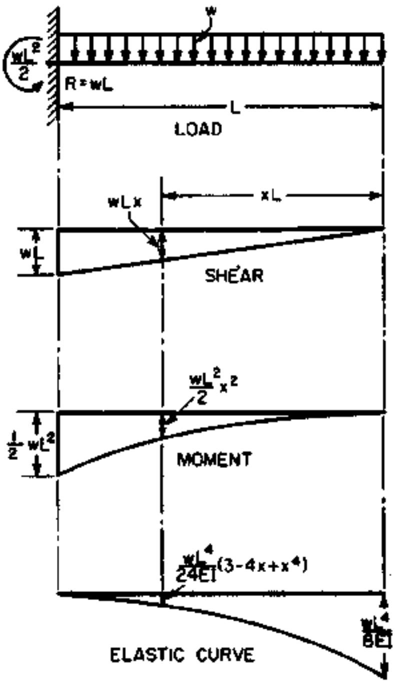


FIGURE 5.39 Uniform load over the whole length of a cantilever.

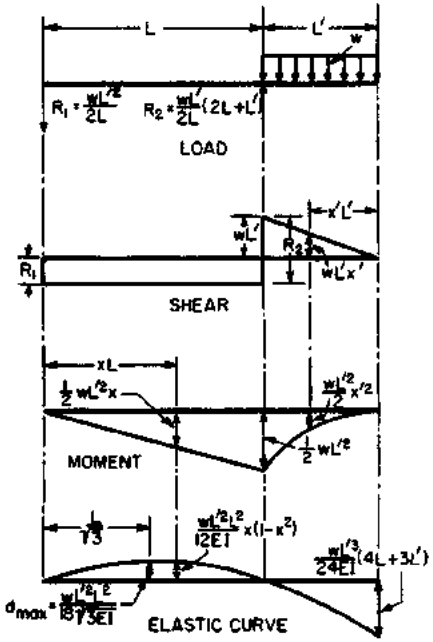


FIGURE 5.40 Uniform load on a beam overhang.

metrical the bending axis coincides with the centroidal axis, whereas in unsymmetrical sections the two axes may be separate. In the latter case, if the plane of the loads contains the centroidal axis but not the bending axis, the beam will be subjected to both bending and torsion.

The **bending axis** may be defined as the longitudinal line in a beam through which transverse loads must pass to preclude the beam's twisting as it bends. The point in each section through which the bending axis passes is called the **shear center,** or center of twist. The shear center is also the center of rotation of the section in pure torsion (Art. 5.4.1).

Computation of stresses and strains in members subjected to both bending and torsion is complicated, because warping of the cross section and buckling effects should be taken into account. Preferably, twisting should be prevented by use of bracing or avoided by selecting appropriate shapes for the members and by locating and directing loads to pass through the bending axis.

(F. Bleich, "Blucking Strength of Metal Structures," McGraw-Hill Publishing Company, New York.)

5.6 CURVED BEAMS

Structural members, such as arches, crane hooks, chain links, and frames of some machines, that have considerable initial curvature in the plane of loading are called

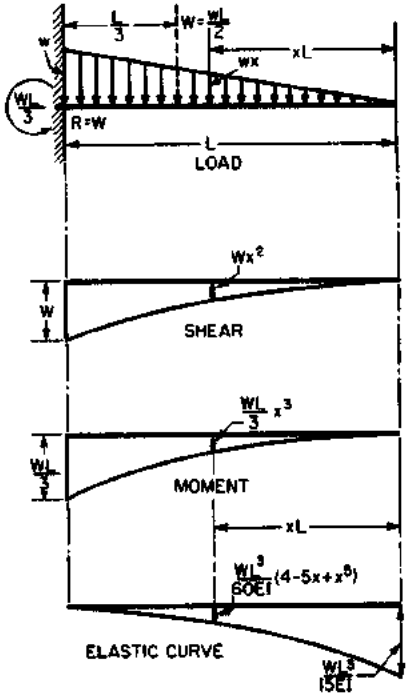


FIGURE 5.41 Triangular loading on a cantilever.

curved beams. The flexure formula of Art. 5.5.10, $f = Mc/I$, cannot be applied to them with any reasonable degree of accuracy unless the depth of the beam is small compared with the radius of curvature.

Unlike the condition in straight beams, unit strains in curved beams are not proportional to the distance from the neutral surface, and the centroidal axis does not coincide with the neutral axis. Hence the stress distribution on a section is not linear but more like the distribution shown in Fig. 5.42*c*.

5.6.1 Stresses in Curved Beams

Just as for straight beams, the assumption that plane sections before bending remain plane after bending generally holds for curved beams. So the total strains are proportional to the distance from the neutral axis. But since the fibers are initially of unequal length, the unit strains are a more complex function of this distance. In Fig. 5.42*a*, for example, the bending couples have rotated section *AB* of the curved beam into section $A'B'$ through an angle $\Delta d\theta$. If ϵ_o is the unit strain at the centroidal axis and ω is the angular unit strain $\Delta d\theta / d\theta$, then the unit strain at a distance y from the centroidal axis (measured positive in the direction of the center of curvature) is

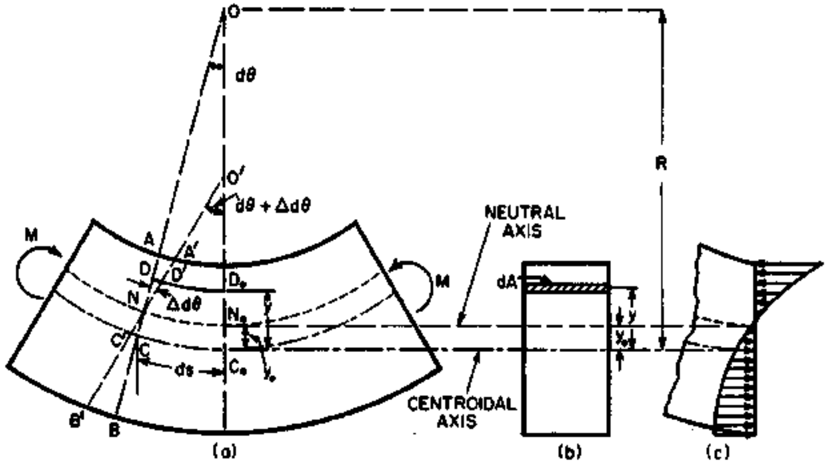


FIGURE 5.42 Bending stresses in a curved beam.

$$\epsilon = \frac{DD'}{DD_o} = \frac{\epsilon_o R\,d\theta - y\Delta d\theta}{(R - y)\,d\theta} = \epsilon_o - (\omega - \epsilon_o)\frac{y}{R - y} \tag{5.75}$$

where R = radius of curvature of centroidal axis.

Equation (5.75) can be expressed in terms of the bending moment if we take advantage of the fact that the sum of the tensile and compressive forces on the section must be zero and the moment of these forces must be equal to the bending moment M. These two equations yield

$$\epsilon_o = \frac{M}{ARE} \qquad \text{and} \qquad \omega = \frac{M}{ARE}\left(1 + \frac{AR^2}{I'}\right) \tag{5.76}$$

where A is the cross-sectional area, E the modulus of elasticity, and

$$I' = \int \frac{y^2\,dA}{1 - y/R} = \int y^2\left(1 + \frac{y}{R} + \frac{y^2}{R^2} + \cdots\right) dA \tag{5.77}$$

It should be noted that I' is very nearly equal to the moment of inertia I about the centroidal axis when the depth of the section is small compared with R, so that the maximum ratio of y to R is small compared with unity. M is positive when it decreases the radius of curvature.

Since the stress $f = E\epsilon$, we obtain the stresses in the curved beam from Eq. (5.75) by multiplying it by E and substituting ϵ_o and ω from Eq. (5.76):

$$f = \frac{M}{AR} - \frac{My}{I'}\,\frac{1}{1 - y/R} \tag{5.78}$$

The distance y_o of the neutral axis from the centroidal axis (Fig. 5.42) may be obtained from Eq. (5.78) by setting $f = 0$:

$$y_o = \frac{I'R}{I' + AR^2} \tag{5.79}$$

Since y_o is positive, the neutral axis shifts toward the center of curvature.

5.6.2 Curved Beams with Various Cross Sections

Equation (5.78) for bending stresses in curved beams subjected to end moments in the plane of curvature can be expressed for the inside and outside beam faces in the form:

$$f = K\frac{Mc}{I} \tag{5.80}$$

where c = distance from the centroidal axis to the inner or outer surface. Table 5.4 gives values of K calculated from Eq. (5.78) for circular, elliptical, and rectangular cross sections.

If Eq. (5.78) is applied to 1 or T beams or tubular members, it may indicate circumferential flange stresses that are much lower than will actually occur. The error is due to the fact that the outer edges of the flanges deflect radially. The effect is equivalent to having only part of the flanges active in resisting bending stresses. Also, accompanying the flange deflections, there are transverse bending stresses in the flanges. At the junction with the web, these reach a maximum, which may be greater than the maximum circumferential stress. Furthermore, there are radial stresses (normal stresses acting in the direction of the radius of curvature) in the web that also may have maximum values greater than the maximum circumferential stress.

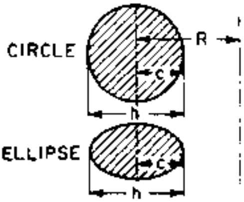
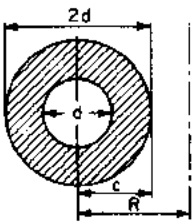
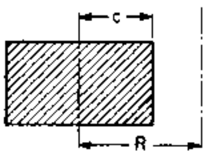
A good approximation to the stresses in I or T beams is as follows: for circumferential stresses, Eq. (5.78) may be used with a modified cross section, which is obtained by using a reduced flange width. The reduction is calculated from $b' = \alpha b$, where b is the length of the portion of the flange projecting on either side from the web, b' is the corrected length, and α is a correction factor determined from equations developed by H. Bleich, α is a function of b^2/rt, where t is the flange thickness and r the radius of the center of the flange:

b^2/rt =	0.5	0.7	1.0	1.5	2	3	4	5
α =	0.9	0.6	0.7	0.6	0.5	0.4	0.37	0.33

When the parameter b^2/rt is greater than 1.0, the maximum transverse bending stress is approximately equal to 1.7 times the stress obtained at the center of the flange from Eq. (5.78) applied to the modified section. When the parameter equals 0.7, that stress should be multiplied by 1.5, and when it equals 0.4, the factor is 1.0 in Eq. (5.78), I' for I beams may be taken for this calculation approximately equal to

$$I' = I\left(1 + \frac{c^2}{R^2}\right) \tag{5.81}$$

TABLE 5.4 Values of K for Curved Beams

Section	$\frac{R}{c}$	K Inside face	K Outside face	y_o
	1.2	3.41	0.54	$0.224R$
	1.4	2.40	0.60	$0.141R$
	1.6	1.96	0.65	$0.108R$
	1.8	1.75	0.68	$0.0847R$
	2.0	1.62	0.71	$0.069R$
	3.0	1.33	0.79	$0.030R$
	4.0	1.23	0.84	$0.016R$
	6.0	1.14	0.89	$0.0070R$
	8.0	1.10	0.91	$0.0039R$
	10.0	1.08	0.93	$0.0025R$
	1.2	3.28	0.58	$0.269R$
	1.4	2.31	0.64	$0.182R$
	1.6	1.89	0.68	$0.134R$
	1.8	1.70	0.71	$0.104R$
	2.0	1.57	0.73	$0.083R$
	3.0	1.31	0.81	$0.038R$
	4.0	1.21	0.85	$0.020R$
	6.0	1.13	0.90	$0.0087R$
	8.0	1.10	0.92	$0.0049R$
	10.0	1.07	0.93	$0.0031R$
	1.2	2.89	0.57	$0.305R$
	1.4	2.13	0.63	$0.204R$
	1.6	1.79	0.67	$0.149R$
	1.8	1.63	0.70	$0.112R$
	2.0	1.52	0.73	$0.090R$
	3.0	1.30	0.81	$0.041R$
	4.0	1.20	0.85	$0.0217R$
	6.0	1.12	0.90	$0.0093R$
	8.0	1.09	0.92	$0.0052R$
	10.0	1.07	0.94	$0.0033R$

where I = moment of inertia of modified section about its centroidal axis
R = radius of curvature of centroidal axis
c = distance from centroidal axis to center of the more sharply curved flange

Because of the high stress factor, it is advisable to stiffen or brace curved I-beam flanges.

The maximum radial stress will occur at the junction of web and flange of I beams. If the moment is negative, that is, if the loads tend to flatten out the beam, the radial stress is tensile, and there is a tendency for the more sharply curved flange to pull away from the web. An approximate value of this maximum stress is

$$f_r = -\frac{A_f}{A}\frac{M}{t_w c_g r'} \tag{5.82}$$

where f_r = radial stress at junction of flange and web of a symmetrical I beam
A_f = area of one flange
A = total cross-sectional area
M = bending moment
t_w = thickness of web
c_g = distance from centroidal axis to center of flange
r' = radius of curvature of inner face of more sharply curved flange

(A. P. Boresi, O. Sidebottom, F. B. Seely, and J. O. Smith, "Advanced Mechanics of Materials," John Wiley & Sons, Inc., New York.)

5.6.3 Axial and Bending Loads on Curved Beams

If a curved beam carries an axial load P as well as bending loads, the maximum unit stress is

$$f = \frac{P}{A} \pm \frac{Mc}{I} K \tag{5.83}$$

where K is a correction factor for the curvature [see Eq. (5.80)]. The sign of M is taken positive in this equation when it increases the curvature, and P is positive when it is a tensile force, negative when compressive.

5.6.4 Slope and Deflection of Curved Beams

If we consider two sections of a curved beam separated by a differential distance ds (Fig. 5.42), the change in angle $\Delta d\theta$ between the sections caused by a bending moment M and an axial load P may be obtained from Eq. (5.76), noting that $d\theta = ds/R$.

$$\Delta d\theta = \frac{M\,ds}{EI'}\left(1 + \frac{I'}{AR^2}\right) + \frac{P\,ds}{ARE} \tag{5.84}$$

where E is the modulus of elasticity, A the cross-sectional area, R the radius of curvature of the centroidal axis, and I' is defined by Eq. (5.77).

If P is a tensile force, the length of the centroidal axis increases by

$$\Delta ds = \frac{P\,ds}{AE} + \frac{M\,ds}{ARE} \tag{5.85}$$

The effect of curvature on shearing deformations for most practical applications is negligible.

For shallow sections (depth of section less than about one-tenth the span), the effect of axial forces on deformations may be neglected. Also, unless the radius of curvature is very small compared with the depth, the effect of curvature may be ignored. Hence, for most practical applications, Eq. (5.84) may be used in the simplified form:

$$\Delta d\theta = \frac{M\,ds}{EI} \tag{5.86}$$

For deeper beams, the action of axial forces, as well as bending moments, should be taken into account; but unless the curvature is sharp, its effect on deformations may be neglected. So only Eq. (5.86) and the first term in Eq. (5.85) need be used. (S. Timoshenko and D. H. Young, "Theory of Structures," McGraw-Hill Publishing Company, New York.) See also Arts. 5.14.1 to 5.14.3.

5.7 BUCKLING OF COLUMNS

Columns are compression members whose cross-sectional dimensions are relatively small compared with their length in the direction of the compressive force. Failure of such members occurs because of instability when a certain axial load P_c (called critical or **Euler load**) is equated or exceeded. The member may bend, or buckle, suddenly and collapse.

Hence the strength P of a column is not determined by the unit stress in Eq. (5.21) ($P = Af$) but by the maximum load it can carry without becoming unstable. The condition of instability is characterized by disproportionately large increases in lateral deformation with slight increase in axial load. Instability may occur in slender columns before the unit stress reaches the elastic limit.

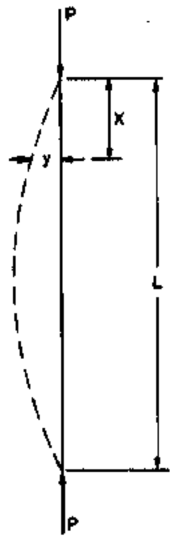


FIGURE 5.43 Buckling of a pin-ended long column.

5.7.1 Stable Equilibrium

Consider, for example, an axially loaded column with ends unrestrained against rotation, shown in Fig. 5.43. If the member is initially perfectly straight, it will remain straight as long as the load P is less than the critical load P_c. If a small transverse force is applied, the column will deflect, but it will return to the straight position when this force is removed. Thus, when P is less than P_c, internal and external forces are in stable equilibrium.

5.7.2 Unstable Equilibrium

If $P = P_c$ and a small transverse force is applied, the column again will deflect, but this time, when the force is removed, the column will remain in the bent position (dashed line in Fig. 5.43). The equation of this elastic curve can be obtained from Eq. (5.62):

$$EI \frac{d^2y}{dx^2} = -P_c y \tag{5.87}$$

in which E = modulus of elasticity
I = least moment of inertia
y = deflection of the bent member from the straight position at a distance x from one end

This assumes, of course, that the stresses are within the elastic limit. Solution of Eq. (5.87) gives the smallest value of the Euler load as

$$P_c = \frac{\pi^2 EI}{L^2} \tag{5.88}$$

Equation (5.88) indicates that there is a definite finite magnitude of an axial load that will hold a column in equilibrium in the bent position when the stresses are below the elastic limit. Repeated application and removal of small transverse forces or small increases in axial load above this critical load will cause the member to fail by buckling. Internal and external forces are in a state of unstable equilibrium.

It is noteworthy that the Euler load, which determines the load-carrying capacity of a column, depends on the stiffness of the member, as expressed by the modulus of elasticity, rather than on the strength of the material of which it is made.

By dividing both sides of Eq. (5.88) by the cross-sectional area A and substituting r^2 for I/A (r is the radius of gyration of the section), we can write the solution of Eq. (5.87) in terms of the average unit stress on the cross section:

$$\frac{P_c}{A} = \frac{\pi^2 E}{(L/r)^2} \tag{5.89}$$

This holds only for the elastic range of buckling; i.e. for values of the slenderness ratio L/r above a certain limiting value that depends on the properties of the material. For inelastic buckling, see Art. 5.7.4.

5.7.3 Effect of End Conditions

Equation (5.89) was derived on the assumption that the ends of the column are free to rotate. It can be generalized, however, to take into account the effect of end conditions:

$$\frac{P_c}{A} = \frac{\pi^2 E}{(kL/r)^2} \tag{5.90}$$

where k is the factor that depends on the end conditions. For a pin-ended column, $k = 1$; for a column with both ends fixed, $k = \frac{1}{2}$; for a column with one end fixed and one end pinned, k is about 0.7; and for a column with one end fixed and one end free from all restraint, $k = 2$.

5.7.4 Inelastic Buckling

Equations (5.88) and (5.90) are derived from Eq. (5.87), the differential equation for the elastic curve. They are based on the assumption that the critical average stress is below the elastic limit when the state of unstable equilibrium is reached. In members with slenderness ratio L/r below a certain limiting value, however, the elastic limit is exceeded before the column buckles. As the axial load approaches the critical load, the modulus of elasticity varies with the stress. Hence Eqs. (5.88) and (5.90), based on the assumption that E is a constant, do not hold for these short columns.

After extensive testing and analysis, prevalent engineering opinion favors the Engesser equation for metals in the inelastic range:

$$\frac{P_t}{A} = \frac{\pi^2 E_t}{(kL/r)^2} \tag{5.91}$$

This differs from Eqs. (5.88) to (5.90) only in that the tangent modulus E_t (the actual slope of the stress-strain curve for the stress P_t/A) replaced the modulus of elasticity E in the elastic range. P_t is the smallest axial load for which two equilibrium positions are possible, the straight position and a deflected position.

5.7.5 Column Curves

Curves obtained by plotting the critical stress for various values of the slenderness ratio are called column curves. For axially loaded, initially straight columns, the column curve consists of two parts: (1) the Euler critical values, and (2) the Engesser, or tangent-modulus critical values.

The latter are greatly affected by the shape of the stress-strain curve for the material of which the column is made, as shown in Fig. 5.44. The stress-strain curve for a material, such as an aluminum alloy or high-strength steel, which does not have a sharply defined yield point, is shown in Fig. 5.44*a*. The corresponding

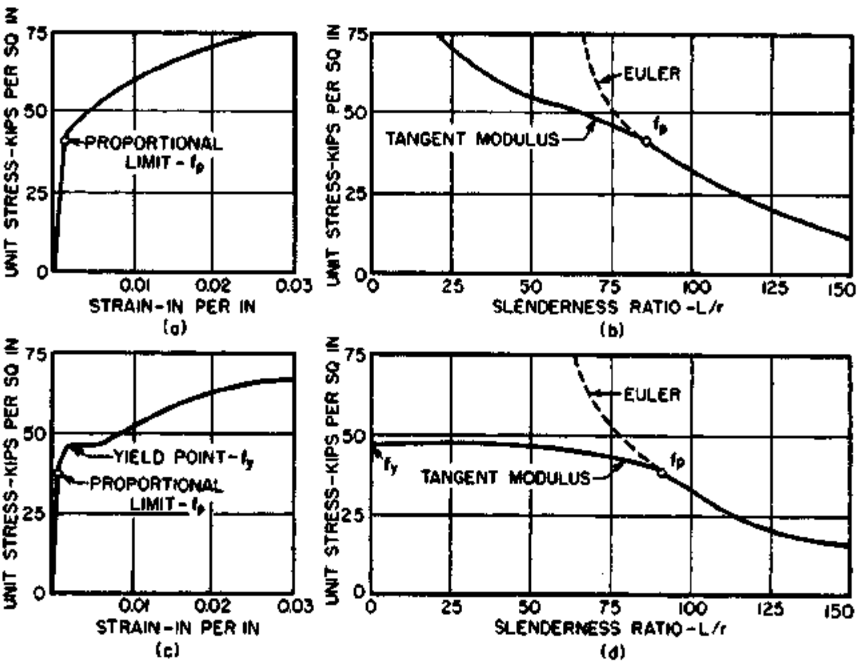


FIGURE 5.44 Column curves: (*a*) stress-strain curve for a material that does not have a sharply defined yield pont: (*b*) column curve for this material; (*c*) stress-strain curve for a material with a sharply defined yield point; (*d*) column curve for that material.

column curve is drawn in Fig. 5.44*b*. In contrast, Fig. 5.44*c* presents the stress-strain curve for structural steel, with a sharply defined point, and Fig. 5.44*d* the related column curve. This curve becomes horizontal as the critical stress approaches the yield strength of the material and the tangent modulus becomes zero, whereas the column curve in Fig. 5.44*b* continues to rise with decreasing values of the slenderness ratio.

Examination of Fig. 44*d* also indicates that slender columns, which fall in the elastic range, where the column curve has a large slope, are very sensitive to variations in the factor k, which represents the effect of end conditions. On the other hand, in the inelastic range, where the column curve is relatively flat, the critical stress is relatively insensitive to changes in k. Hence the effect of end conditions on the stability of a column is of much greater significance for long columns than for short columns.

5.7.6 Local Buckling

A column may not only fail by buckling of the member as a whole but as an alternative, by buckling of one of its components. Hence, when members like I beams, channels, and angles are used as columns or when sections are built up of plates, the possibility of the critical load on a component (leg, half flange, web, lattice bar) being less than the critical load on the column as a whole should be investigated.

Similarly, the possibility of buckling of the compression flange or the web of a beam should be looked into.

Local buckling, however, does not always result in a reduction in the load-carrying capacity of a column. Sometimes, it results in a redistribution of the stresses enabling the member to carry additional load.

5.7.7 Behavior of Actual Columns

For many reasons, columns in structures behave differently from the ideal column assumed in deriving Eqs. (5.88) and (5.91). A major consideration is the effect of accidental imperfections, such as nonhomogeneity of materials, initial crookedness, and unintentional eccentricities of the axial load, since neither field nor shopwork can be perfect. These and the effects of residual stresses usually are taken into account by a proper choice of safety factor.

There are other significant conditions, however, that must be considered in any design rule: continuity in frame structures and eccentricity of the axial load. Continuity affects column action in two ways. The restraint at column ends determines the value of k, and bending moments are transmitted to the column by adjoining structural members.

Because of the deviation of the behavior of actual columns from the ideal, columns generally are designed by empirical formulas. Separate equations usually are given for short columns, intermediate columns, and long columns. For specific materials—steel, concrete, timber—these formulas are given in Secs. 7 to 10.

For more details on column action, see F. Bleich, "Buckling Strength of Metal Structures," McGraw-Hill Publishing Company, New York, 1952: S. Timoshenko and J. M. Gere, "Theory of Elastic Stability," McGraw-Hill Publishing Company, New York, 1961; and T. V. Galambos, "Guide to Stability Design Criteria for Metal Structures," 4th ed., John Wiley & Sons, Inc., Somerset, N.J., 1988.

5.8 GRAPHIC-STATICS FUNDAMENTALS

A force may be represented by a straight line of fixed length. The length of line to a given scale represents the magnitude of the force. The position of the line parallels the line of action of the force. And an arrowhead on the line indicates the direction in which the force acts.

Forces are concurrent when their lines of action meet. If they lie in the same plane, they are coplanar.

5.8.1 Parallelogram of Forces

The resultant of several forces is a single forces that would produce the same effect on a rigid body. The resultant of two concurrent forces is determined by the parallelogram law:

If a parallelogram is constructed with two forces as sides, the diagonal represents the resultant of the forces (Fig. 5.45*a*).

The **resultant** is said to be equal to the sum of the forces, sum here meaning, of course, addition by the parallelogram law. Subtraction is carried out in the same manner as addition, but the direction of the force to be subtracted is reversed.

If the direction of the resultant is reversed, it becomes the **equilibrant,** a single force that will hold the two given forces in equilibrium.

5.8.2 Resolution of Forces

To resolve a force into two components, a parallelogram is drawn with the force as a diagonal. The sides of the parallelogram represent the components. The procedure is: (1) Draw the given force. (2) From both ends of the force draw lines parallel to the directions in which the components act. (3) Draw the components along the parallels through the origin of the given force to the intersections with the parallels through the other end. Thus, in Fig. 5.45*a*, P_1 and P_2 are the components in directions *OA* and *OB* of the force represented by *OC*.

5.8.3 Force Polygons

Examination of Fig. 5.45*a* indicates that a step can be saved in adding the two forces. The same resultant could be obtained by drawing only the upper half of the parallelogram. Hence, to add two forces, draw the first force; then draw the second

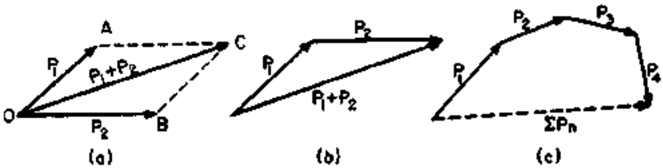


FIGURE 5.45 Addition of forces by (*a*) parallelogram law; (*b*) triangle construction; (*c*) polygon construction.

force beginning at the end of the first one. The resultant is the force drawn from the origin of the first force to the end of the second force, as shown in Fig. 5.45*b*. Again, the equilibrant is the resultant with direction reversed.

From this diagram, an important conclusion can be drawn: **If three forces meeting at a point are in equilibrium, they will form a closed force triangle.**

The conclusions reached for addition of two forces can be generalized for several concurrent forces: To add several forces, P_1, P_2, P_3, . . . , P_n, draw P_2 from the end of P_1, P_3 from the end of P_2, etc. The force required to close the force polygon is the resultant (Fig. 5.45*c*).

If a group of concurrent forces are in equilibrium, they will form a closed force polygon.

5.9 ROOF TRUSSES

A truss is a coplanar system of structural members joined together at their ends to form a stable framework. If small changes in the lengths of the members due to loads are neglected, the relative positions of the joints cannot change.

5.9.1 Characteristics of Trusses

Three bars pinned together to form a triangle represents the simplest type of truss. Some of the more common types of roof trusses are shown in Fig. 6.46.

The top members are called the upper chord; the bottom members, the lower chord; and the verticals and diagonals web members.

The purpose of roof trusses is to act like big beams, to support the roof covering over long spans. They not only have to carry their own weight and the weight of the roofing and roof beams, or purlins, but cranes, wind loads, snow loads, suspended ceilings, and equipment, and a live load to take care of construction, maintenance, and repair loading. These loads are applied at the intersection of the members, or panel points, so that the members will be subjected principally to axial stresses—tension or compression.

Methods of computing stresses in trusses are presented in Arts. 5.9.3 and 5.9.4. A method of computing truss deflections is described in Art. 5.10.4.

5.9.2 Bow's Notation

For simple designation of loads and stresses, capital letters are placed in the spaces between truss members and between forces. Each member and load is then designated by the letters on opposite sides of it. For example, in Fig. 5.47*a*, the upper chord members are *AF*, *BH*, *CJ*, and *DL*. The loads are *AB*, *BC*, and *CD*, and the reactions are *EA* and *DE*. Stresses in the members generally are designated by the same letters but in lowercase.

5.9.3 Method of Joints

A useful method for determining the stresses in truss members is to select sections that isolate the joints one at a time and then apply the laws of equilibrium to each.

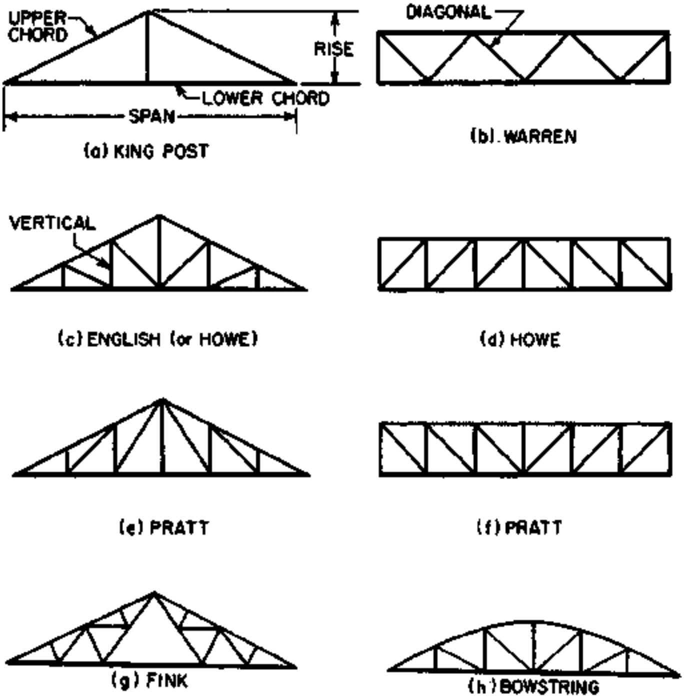


FIGURE 5.46 Common types of roof trusses.

Considering the stresses in the cut members as external forces, the sum of the horizontal components of the forces acting at a joint must be zero, and so must be the sum of the vertical components. Since the lines of action of all the forces are known, we can therefore compute two unknown magnitudes at each joint by this method. The procedure is to start at a joint that has only two unknowns (generally at the support) and then, as stresses in members are determined, analyze successive joints.

Let us, for illustration, apply the method to joint 1 of the truss in Fig. 5.47*a*. Equating the sum of the vertical components to zero, we find that the vertical component of the top-chord must be equal and opposite to the reaction, 12 kips (12,000 lb). The stress in the top chord at this joint, then, must be a compression equal to $12 \times {}^{30}\!/_{18} = 20$ kips. From the fact that the sum of the horizontal components must be zero, we find that the stress in the bottom chord at the joint must be equal and opposite to the horizontal component of the top chord. Hence the stress in the bottom chord must be a tension equal to $20 \times {}^{24}\!/_{30} = 16$ kips.

Moving to joint 2, we note that, with no vertical loads at the joint, the stress in the vertical is zero. Also, the stress is the same in both bottom chord members at the joint, since the sum of the horizontal components must be zero.

Joint 3 now contains only two unknown stresses. Denoting the truss members and the loads by the letters placed on opposite sides of them, as indicated in Fig. 5.47*a*, the unknown stresses are S_{BH} and S_{HG}. The laws of equilibrium enable us to

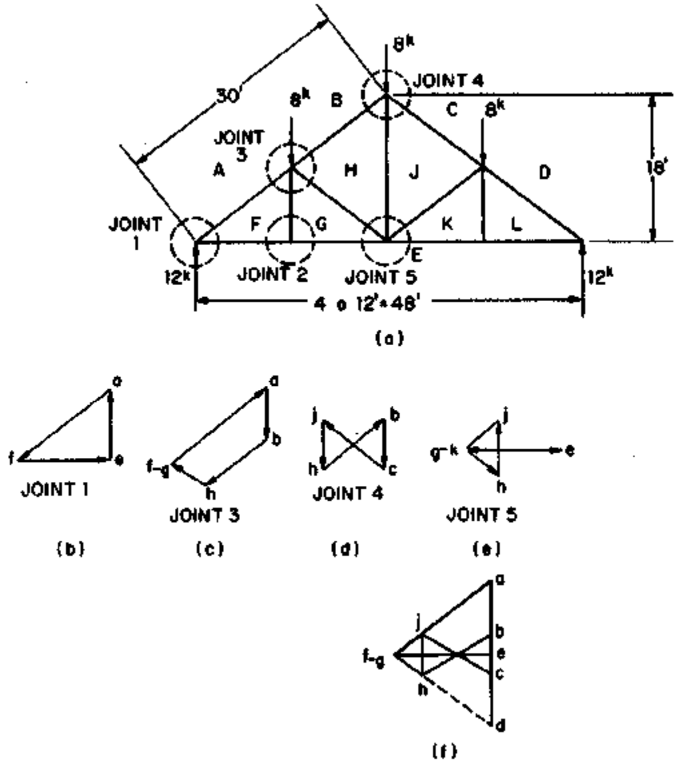


FIGURE 5.47 Method of joints applied to the roof truss shown in (*a*). Stresses in members at each joint are determined graphically in sucession (*b*) to (*e*).

write the following two equations, one for the vertical components and the second for the horizontal components:

$$\Sigma V = 0.6S_{FA} - 8 - 0.6S_{BH} + 0.6S_{HG} = 0$$

$$\Sigma H = 0.8S_{FA} - 0.8S_{BH} - 0.8S_{HG} = 0$$

Both unknown stresses are assumed to be compressive; i.e., acting toward the joint. The stress in the vertical does not appear in these equations, because it was already determined to be zero. The stress in *FA*, S_{FA}, was found from analysis of joint 1 to be 20 kips. Simultaneous solution of the two equations yields $S_{HG} = 6.7$ kips and $S_{BH} = 13.3$ kips. (If these stresses had come out with a negative sign, it would have indicated that the original assumption of their directions was incorrect; they would, in that case, be tensile forces instead of compressive forces.) See also Art. 5.9.4.

All the force polygons in Fig. 5.47 can be conveniently combined into a single stress diagram. The combination (Fig. 5.47*f*) is called a **Maxwell diagram.**

5.9.4 Method of Sections

An alternative method to that described in Art. 5.9.3 for determining the stresses in truss members is to isolate a portion of the truss by a section so chosen as to cut only as many members with unknown stresses as can be evaluated by the laws of equilibrium applied to that portion of the truss. The stresses in the cut members are treated as external forces. Compressive forces act toward the panel point and tensile forces away from the joint.

Suppose, for example, we wish to find the stress in chord *AB* of the truss in Fig. 5.48*a*. We can take a vertical section *XX* close to panel point *A*. This cuts not only *AB* but *AD* and *ED* as well. The external 10-kip (10,000-lb) loading and 25-kip reaction at the left are held in equilibrium by the compressive force *C* in *AB*, tensile force *T* in *ED*, and tensile force *S* in *AD* (Fig. 5.48*b*). The simplest way to find *C* is to take moments about *D*, the point of intersection of *S* and *T*, eliminating these unknowns from the calculation.

$$-9C + 36 \times 25 - 24 \times 10 - 12 \times 10 = 0$$

from which *C* is found to be 60 kips.

Similarly, to find the stress in *ED*, the simplest way is to take moments about *A*, the point of intersection of *S* and *C*:

$$-9T + 24 \times 25 - 12 \times 10 = 0$$

from which *T* is found to be 53.3 kips.

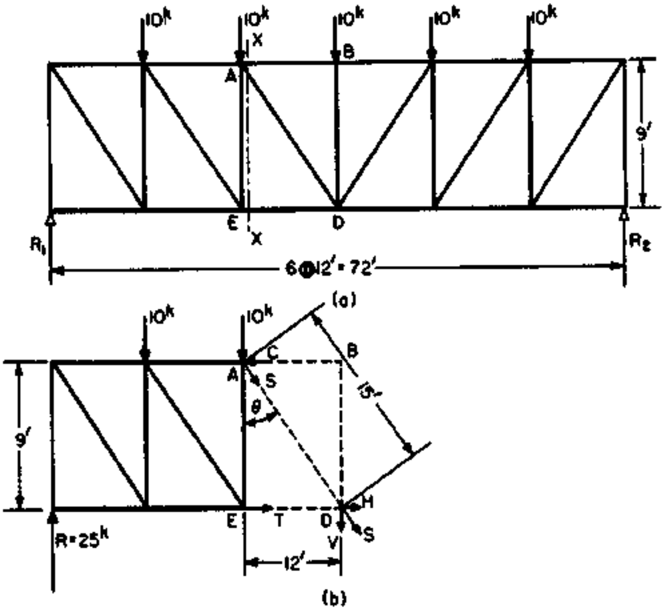


FIGURE 5.48 Stresses in truss members cut by section *XX*, shown in (*a*), are determined by method of sections (*b*).

On the other hand, the stress in *AD* can be easily determined by two methods. One takes advantage of the fact that *AB* and *ED* are horizontal members, requiring *AD* to carry the full vertical shear at section *XX*. Hence we know that the vertical component V of $S = 25 - 10 - 10 = 5$ kips. Multiplying V by sec θ (Fig. 5.48*b*), which is equal to the ratio of the length of *AD* to the rise of the truss ($^{15}/_{9}$), S is found to be 8.3 kips. The second method—presented because it is useful when the chords are not horizontal—is to resolve S into horizontal and vertical components at D and take moments about E. Since both T and the horizontal component of S pass through E, they do not appear in the computations, and C already has been computed. Equating the sum of the moments to zero gives $V = 5$, as before.

Some trusses are complex and require special methods of analysis. (Norris et al., ''Elementary Structural Analysis,'' 4th ed., McGraw-Hill Book Company, New York).

5.10 GENERAL TOOLS FOR STRUCTURAL ANALYSIS

For some types of structures, the equilibrium equations are not sufficient to determine the reactions or the internal stresses. These structures are called **statically indeterminate.**

For the analysis of such structures, additional equations must be written on the basis of a knowledge of the elastic deformations. Hence methods of analysis that enable deformations to be evaluated in terms of unknown forces or stresses are important for the solution of problems involving statically indeterminate structures. Some of these methods, like the method of virtual work, are also useful in solving complicated problems involving statically determinate systems.

5.10.1 Virtual Work

A virtual displacement is an imaginary small displacement of a particle consistent with the constraints upon it. Thus, at one support of a simply supported beam, the virtual displacement could be an infinitesimal rotation $d\theta$ of that end but not a vertical movement. However, if the support is replaced by a force, then a vertical virtual displacement may be applied to the beam at that end.

Virtual work is the product of the distance a particle moves during a virtual displacement by the component in the direction of the displacement of a force acting on the particle. If the displacement and the force are in opposite directions, the virtual work is negative. When the displacement is normal to the force, no work is done.

Suppose a rigid body is acted upon by a system of forces with a resultant R. Given a virtual displacement ds at an angle α with R, the body will have virtual work done on it equal to $R \cos \alpha \, ds$. (No work is done by internal forces. They act in pairs of equal magnitude but opposite direction, and the virtual work done by one force of a pair is equal but opposite in sign to the work done by the other force.) If the body is in equilibrium under the action of the forces, then $R = 0$ and the virtual work also is zero.

Thus, the principle of virtual work may be stated: **If a rigid body in equilibrium is given a virtual displacement, the sum of the virtual work of the forces acting on it must be zero.**

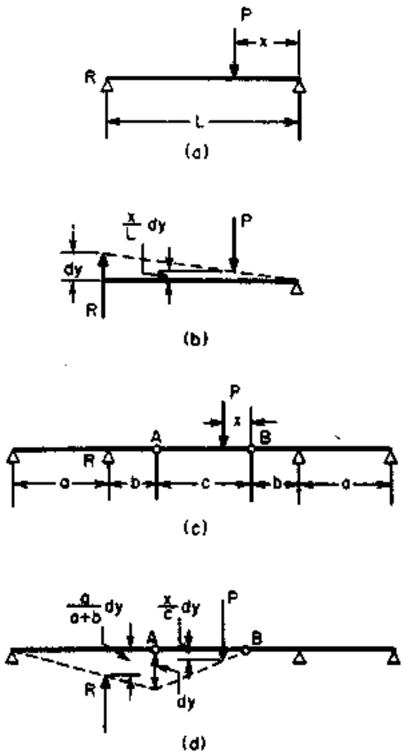


FIGURE 5.49 Principle of virtual work applied to determination of a simple-beam reaction (*a*) and (*b*) and to the reaction of a beam with a suspended span (*c*) and (*d*).

As an example of how the principle may be used to find a reaction of a statically determinate beam, consider the simple beam in Fig. 5.49*a*, for which the reaction R is to be determined. First, replace the support by an unknown force R. Next, move that end of the beam upward a small amount dy as in Fig. 5.49*b*. The displacement under the load P will be $x\ dy/L$, upward. Then, by the principle of virtual work, $R\ dy - Px\ dy/L = 0$, from which $R = Px/L$.

The principle may also be used to find the reaction R of the more complex beam in Fig. 5.49*c*. The first step again is to replace the support by an unknown force R. Next, apply a virtual downward displacement dy at hinge A (Fig. 5.49*d*). Displacement under load P is $x\ dy/c$, and at the reaction R, $a\ dy/(a + b)$. According to the principle of virtual work, $-Ra\ dy/(a + b) + Px\ dy/c = 0$, from which reaction $R = Px(a + b)/ac$. In this type of problem, the method has the advantage that only one reaction need be considered at a time and internal forces are not involved.

5.10.2 Strain Energy

When an elastic body is deformed, the virtual work done by the internal forces is equal to the corresponding increment of the strain energy dU, in accordance with the principle of virtual work.

Assume a constrained elastic body acted upon by forces $P_1, P_2, \ldots,$ for which the corresponding deformations are $e_1, e_2 \ldots$. Then, $\Sigma P_n\ de_n = dU$. The increment of the strain energy due to the increments of the deformations is given by

$$dU = \frac{\partial U}{\partial e_1}\, de_1 + \frac{\partial U}{\partial e_2}\, de_2 + \cdots$$

In solving a specific problem, a virtual displacement that is not convenient in simplifying the solution should be chosen. Suppose, for example, a virtual displacement is selected that affects only the deformation e_n corresponding to the load P_n, other deformations being unchanged. Then, the principle of virtual work requires that

$$P_n\, de_n = \frac{\partial U}{\partial e_n}\, de_n$$

This is equivalent to

$$\frac{\partial U}{\partial e_n} = P_n \tag{5.92}$$

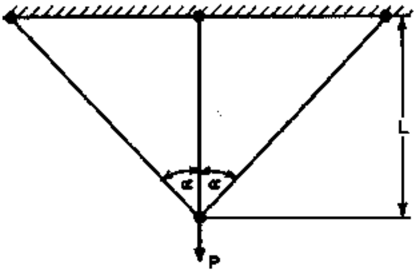


FIGURE 5.50 Statically indeterminate truss.

which states that the partial derivative of the strain energy with respect to any specific deformation gives the corresponding force.

Suppose, for example, the stress in the vertical bar in Fig. 5.50 is to be determined. All bars are made of the same material and have the same cross section. If the vertical bar stretches an amount e under the load P, the inclined bars will each stretch an amount $e \cos \alpha$. The strain energy in the system is [from Eq. (5.30)]

$$U = \frac{AE}{2L}(e^2 + 2e^2 \cos^3 \alpha)$$

and the partial derivative of this with respect to e must be equal to P; that is

$$P = \frac{AE}{2L}(2e + 4e \cos^3 \alpha)$$

$$= \frac{AEe}{L}(1 + 2 \cos^3 \alpha)$$

Noting that the force in the vertical bar equals AEe/L, we find from the above equation that the required stress equals $P/(1 + 2 \cos^3 \alpha)$.

Castigliano's Theorems. It can also be shown that, if the strain energy is expressed as a function of statically independent forces, the partial derivative of the strain energy with respect to one of the forces gives the deformation corresponding to that force. (See Timoshenko and Young, "Theory of Structures," McGraw-Hill Publishing Company, New York.)

$$\frac{\partial U}{\partial P_n} = e_n \tag{5.93}$$

This is known as Castigliano's first theorem. (His second theorem is the principle of least work.)

5.10.3 Method of Least Work

If displacement of a structure is prevented, as at a support, the partial derivative of the strain energy with respect to that supporting force must be zero, according to Castigliano's first theorem. This establishes his second theorem:

The strain energy in a statically indeterminate structure is the minimum consistent with equilibrium.

As an example of the use of the method of least work, we shall solve again for the stress in the vertical bar in Fig. 5.50. Calling this stress X, we note that the stress in each of the inclined bars must be $(P - X)/2 \cos \alpha$. With the aid of Eq. (5.30), we can express the strain energy in the system in terms of X as

$$U = \frac{X^2L}{2AE} + \frac{(P - X)^2L}{4AE \cos^3 \alpha}$$

Hence, the internal work in the system will be a minimum when

$$\frac{\partial U}{\partial X} = \frac{XL}{AE} - \frac{(P - X)L}{2AE \cos^3 \alpha} = 0$$

Solving for X gives the stress in the vertical bar as $P/(1 + 2 \cos^3 \alpha)$, as before (Art. 5.10.1).

5.10.4 Dummy Unit-Load Method

In Art. 5.2.7, the strain energy for pure bending was given as $U = M^2L/2EI$ in Eq. (5.33). To find the strain energy due to bending stress in a beam, we can apply this equation to a differential length dx of the beam and integrate over the entire span. Thus,

$$U = \int_0^L \frac{M^2\,dx}{2EI} \tag{5.94}$$

If M represents the bending moment due to a generalized force P, the partial derivative of the strain energy with respect to P is the deformation d corresponding to P. Differentiating Eq. (5.94) under the integral sign gives

$$d = \int_0^L \frac{M}{EI} \frac{\partial M}{\partial P}\,dx \tag{5.95}$$

The partial derivative in this equation is the rate of change of bending moment with the load P. It is equal to the bending moment m produced by a unit generalized load applied at the point where the deformation is to be measured and in the direction of the deformation. Hence, Eq. (5.95) can also be written

$$d = \int_0^L \frac{Mm}{EI}\,dx \tag{5.96}$$

To find the vertical deflection of a beam, we apply a vertical dummy unit load at the point where the deflection is to be measured and substitute the bending moments due to this load and the actual loading in Eq. (5.96). Similarly, to compute a rotation, we apply a dummy unit moment.

Beam Deflections. As a simple example, let us apply the dummy unit-load method to the determination of the deflection at the center of a simply supported, uniformly loaded beam of constant moment of inertia (Fig. 5.51*a*). As indicated in Fig. 5.51*b*, the bending moment at a distance x from one end is $(wL/2)x - (w/2)x^2$. If we apply a dummy unit load vertically at the center of the beam (Fig.

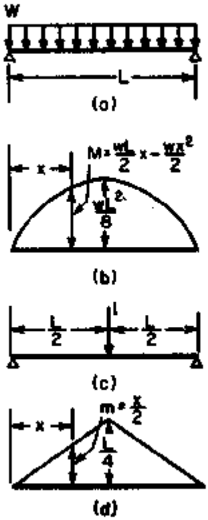


FIGURE 5.51 Dummy unit-load method applied to a uniformly loaded, simple beam (*a*) to find mid-span deflection; (*b*) moment diagram for the uniform load; (*c*) unit load at midspan: (*d*) moment diagram for the unit load.

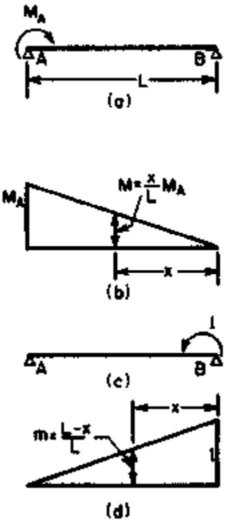


FIGURE 5.52 End rotation of a simple beam due to an end moment: (*a*) by dummy unit-load method; (*b*) moment diagram for the end moment; (*c*) unit moment applied at beam end; (*d*) moment diagram for the unit moment.

5.51*c*), where the vertical deflection is to be determined, the moment at x is $x/2$, as indicated in Fig. 5.51*d*. Substituting in Eq. (5.96) and taking advantage of the symmetry of the loading gives

$$d = 2\int_0^{L/2}\left(\frac{wL}{2}x - \frac{w}{2}x^2\right)\frac{x}{2}\frac{dx}{EI} = \frac{5wL^4}{384EI}$$

Beam End Rotations. As another example, let us apply the method to finding the end rotation at one end of a simply supported, prismatic beam produced by a moment applied at the other end. In other words, the problem is to find the end rotation at B, θ_B, in Fig. 5.52*a*, due to M_A. As indicated in Fig. 5.52*b*, the bending moment at a distance x from B caused by M_A is $M_A x/L$. If we applied a dummy unit moment at B (Fig. 5.52*c*), it would produce a moment at x of $(L - x)/L$ (Fig. 5.52*d*). Substituting in Eq. (5.96) gives

$$\theta_B = \int_0^L M_A \frac{x}{L}\frac{L - x}{L}\frac{dx}{EI} - \frac{M_A L}{6EI}$$

Shear Deflections. To determine the deflection of a beam caused by shear, Castigliano's theorems can be applied to the strain energy in shear

$$V = \int\int \frac{v^2}{2G}\,dA\,dx$$

where v = shearing unit stress
G = modulus of rigidity
A = cross-sectional area

Truss Deflections. The dummy unit-load method may also be adapted for the determination of the deformation of trusses. As indicated by Eq. (5.30), the strain energy in a truss is given by

$$U = \sum \frac{S^2L}{2AE} \tag{5.97}$$

which represents the sum of the strain energy for all the members of the truss. S is the stress in each member caused by the loads. Applying Castigliano's first theorem and differentiating inside the summation sign yield the deformation:

$$d = \sum \frac{SL}{AE} \frac{\partial S}{\partial P} \tag{5.98}$$

The partial derivative in this equation is the rate of change of axial stress with the load P. It is equal to the axial stress u in each bar of the truss produced by a unit load applied at the point where the deformation is to be measured and in the direction of the deformation. Consequently, Eq. (5.98) can also be written

$$d = \sum \frac{Sul}{AE} \tag{5.99}$$

To find the deflection of a truss, apply a vertical dummy unit load at the panel point where the deflection is to be measured and substitute in Eq. (5.99) the stresses in each member of the truss due to this load and the actual loading. Similarly, to find the rotation of any joint, apply a dummy unit moment at the joint, compute the stresses in each member of the truss, and substitute in Eq. (5.99). When it is necessary to determine the relative movement of two panel points, apply dummy unit loads in opposite directions at those points.

It is worth noting that members that are not stressed by the actual loads or the dummy loads do not enter into the calculation of a deformation.

As an example of the application of Eq. (5.99), let us compute the deflection of the truss in Fig. 5.53. The stresses due to the 20-kip load at each panel point are shown in Fig. 5.53*a*, and the ratio of length of members in inches to their cross-sectional area in square inches is given in Table 5.5. We apply a vertical dummy unit load at L_2, where the deflection is required. Stresses u due to this load are shown in Fig. 5.53*b* and Table 5.5.

The computations for the deflection are given in Table 5.5. Members not stressed by the 20-kip loads or the dummy unit load are not included. Taking advantage of the symmetry of the truss, we tabulate the values for only half the truss and double the sum.

$$d = \frac{SuL}{AE} = \frac{2 \times 13.742{,}000}{30{,}000{,}000} = 0.916 \text{ in}$$

Also, to reduce the amount of calculation, we do not include the modulus of elasticity E, which is equal to 30,000,000, until the very last step, since it is the same for all members.

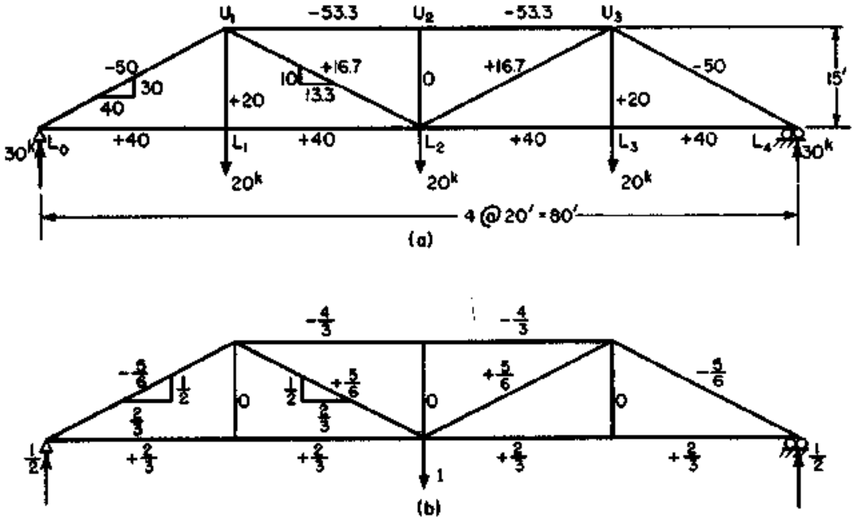


FIGURE 5.53 Dummy unit-load method applied to the loaded truss shown in (*a*) to find midspan deflection; (*b*) unit load applied at midspan.

TABLE 5.5 Deflection of a Truss

Member	L/A	S	u	SuL/A
L_0L_2	160	+40	$+2/3$	4,267
L_0U_1	75	−50	$-5/6$	3,125
U_1U_2	60	−53.3	$-4/3$	4,267
U_1L_2	150	+16.7	$+5/6$	2,083
				13,742

5.10.5 Reciprocal Theorem and Influence Lines

Consider a structure loaded by a group of independent forces A, and suppose that a second group of forces B are added. The work done by the forces A acting over the displacements due to B will be W_{AB}.

Now, suppose the forces B had been on the structure first, and then load A had been applied. The work done by the forces B acting over the displacements due to A will be W_{BA}.

The reciprocal theorem states that $W_{AB} = W_{BA}$.

Some very useful conclusions can be drawn from this equation. For example, there is the reciprocal deflection relationship: **The deflection at a point *A* due to a load at *B* is equal to the deflection at *B* due to the same load applied at *A*. Also, the rotation at *A* due to a load (or moment) at *B* is equal to the rotation at *B* due to the same load (or moment) applied at *A*.**

Another consequence is that deflection curves may also be influence lines to some scale for reactions, shears, moments, or deflections **(Muller-Breslau principles).** (Influence lines are defined in Art. 5.5.8.) For example, suppose the influence line for a reaction is to be found; that is, we wish to plot the reaction R as a unit load moves over the structure, which may be statically indeterminate. For the loading condition A, we analyze the structure with a unit load on it at a distance x from some reference point. For loading condition B, we apply a dummy unit vertical load upward at the place where the reaction is to be determined, deflecting the structure off the support. At a distance x from the reference point, the displacement in d_{xR} and over the support the displacement is d_{RR}. Hence $W_{AB} = -1\ (D_{xR}) + Rd_{RR}$. On the other hand, W_{BA} is zero, since loading condition A provides no displacement for the dummy unit load at the support in condition B. Consequently, from the reciprocal theorem,

$$R = \frac{d_{xR}}{d_{RR}}$$

Since d_{RR} is a constant, R is proportional to d_{xR}. Hence the influence line for a reaction can be obtained from the deflection curve resulting from a displacement of the support (Fig. 5.54). The magnitude of the reaction is obtained by dividing each ordinate of the deflection curve by the displacement of the support.

Similarly, the influence line for shear can be obtained from the deflection curve produced by cutting the structure and shifting the cut ends vertically at the point for which the influence line is desired (Fig. 5.55).

The influence line for bending moment can be obtained from the deflection curve produced by cutting the structure and rotating the cut ends at the point for which the influence line is desired (Fig. 5.56).

And finally, it may be noted that the deflection curve for a load of unity at some point of a structure is also the influence line for deflection at that point (Fig. 5.57).

5.10.6 Superposition Methods

The principle of superposition applies when the displacement (deflection or rotation) of every point of a structure is directly proportional to the applied loads. The

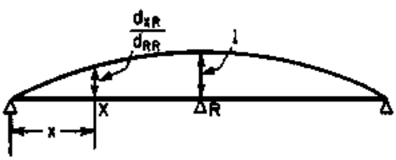


FIGURE 5.54 Reaction-influence line for a continuous beam.

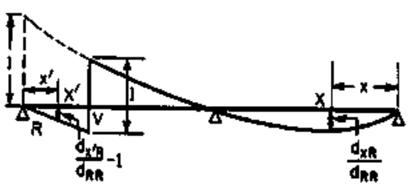


FIGURE 5.55 Shear-influence line for a continuous beam.

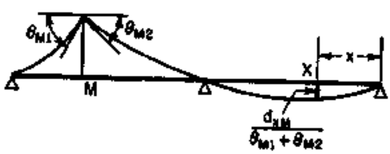


FIGURE 5.56 Moment-influence line for a continuous beam.

FIGURE 5.57 Deflection-influence line for a continuous beam.

principle states that the displacement at each point caused by several loads equals the sum of the displacements at the point when the loads are applied to the structure individually in any sequence. Also, the bending moment (or shear) at every point induced by applied loads equals the sum of the bending moments (or shears) induced at the point by the loads applied individually in any sequence.

The principle holds for linearly elastic structures, for which unit stresses are proportional to unit strains, when displacements are very small and calculations can be based on the underformed configuration of the structure without significant error.

As a simple example, consider a bar with length L and cross-sectional area A loaded with n axial loads $P_1, P_2 \ldots P_n$. Let F equal the sum of the loads. From Eq. (5.23), F causes an elongation $\delta = FL/AE$, where E is the modulus of elasticity of the bar. According to the principle of superposition, if e_1 is the elongation caused by P_1 alone, e_2 by P_2 alone, . . and e_n by P_n alone, then regardless of the sequence in which the loads are applied, when all the loads are on the bar,

$$\delta = e_1 + e_2 + \cdots + e_n$$

This simple case can be easily verified by substituting $e_1 = P_1L/AE$, $e_2 = P_2L/AE$, . . . , and $e_n = P_nL/AE$ in this equation and noting that $F = P_1 + P_2 + \cdots + P_n$:

$$\delta = \frac{P_1L}{AE} + \frac{P_2L}{AE} + \cdots + \frac{P_nL}{AE} = (P_1 + P_2 + \cdots + P_n)\frac{L}{AE} = \frac{FL}{AE}$$

In the preceding equations, L/AE represents the elongation induced by a unit load and is called the **flexibility** of the bar.

The reciprocal, AE/L, represents the force that causes a unit elongation and is called the **stiffness** of the bar.

Analogous properties of beams, columns, and other structural members and the principle of superposition are useful in analysis of many types of structures. Calculation of stresses and displacements of statically indeterminate structures, for example, often can be simplified by resolution of bending moments, shears, and displacements into components chosen to supply sufficient equations for the solution from requirements for equilibrium of forces and compatibility of displacements.

Consider the continuous beam *ALRBC* shown in Fig. 5.58*a*. Under the loads shown, member *LR* is subjected to end moments M_L and M_R (Fig. 5.58*b*) that are initially unknown. The bending-moment diagram for *LR* for these end moments is shown at the left in Fig. 5.58*c*. If these end moments were known, *LR* would be statically determinate; that is *LR* could be treated as a simply supported beam subjected to known end moments M_L and M_R. The analysis can be further simplified by resolution of the bending-moment diagram into the three components shown to the right of the equal sign in Fig. 5.58*c*. This example leads to the following conclusion:

The bending moment at any section of a span *LR* of a continuous beam or frame equals the simple-beam moment due to the applied loads, plus the simple-beam moment due to the end moment at *L*, plus the simple-beam moment due to the end moment at *R*.

When the moment diagrams for all the spans of *ALRBC* in Fig. 5.58 have been resolved into components so that the spans may be treated as simple beams, all the end moments (moments at supports) can be determined from two basic requirements:

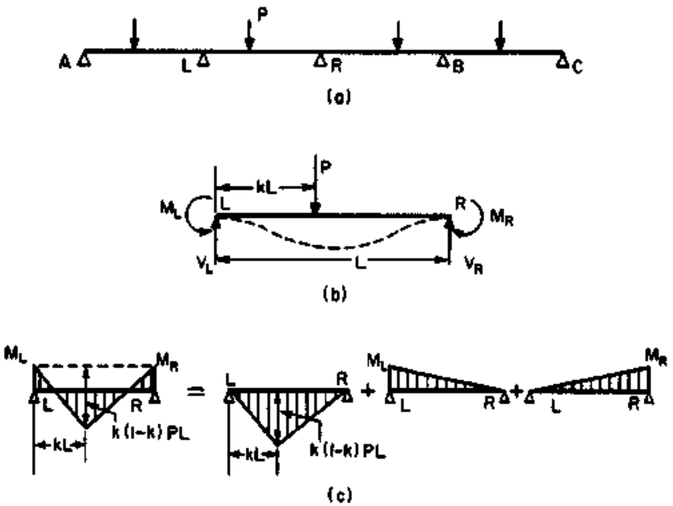


FIGURE 5.58 Any span of a continuous beam (*a*) can be treated as a simple beam, as shown in (*b*) and (*c*), the moment diagram is resolved into basic components.

1. The sum of the moments at every support equals zero.
2. The end rotation (angular change at the support) of each member rigidly connected at the support is the same.

5.10.7 Influence-Coefficient Matrices

A matrix is a rectangular array of numbers in rows and columns that obeys certain mathematical rules known generally as matrix algebra and matrix calculus. A matrix consisting of only a single column is called a **vector.** In this book, matrices and vectors are represented by boldfaced letters and their elements by lightface symbols, with appropriate subscripts. It often is convenient to use numbers for the subscripts to indicate the position of an element in the matrix. Generally, the first digit indicates the row and the second digit the column. Thus, in matrix **A**, A_{23} represents the element in the second row and third column:

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \tag{5.100}$$

Methods based on matrix representations often are advantageous for structural analysis and design of complex structures. One reason is that matrices provide a compact means of representing and manipulating large quantities of numbers. Another reason is that computers can perform matrix operations automatically and speedily. Computer programs are widely available for this purpose.

Matrix Equations. Matrix notation is especially convenient in representing the solution of simultaneous liner equations, which arise frequently in structural analysis. For example, suppose a set of equations is represented in matrix notation by

$\mathbf{AX} = \mathbf{B}$, where $\mathbf{X}$ is the vector of variables $X_1, X_2, \ldots, X_n$, $\mathbf{B}$ is the vector of the constants on the right-hand side of the equations, and $\mathbf{A}$ is a matrix of the coefficients of the variables. Multiplication of both sides of the equation by $\mathbf{A}^{-1}$, the inverse of $\mathbf{A}$, yields $\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{B}$. Since $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$, the identity matrix, and $\mathbf{IX} = \mathbf{X}$, the solution of the equations is represented by $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$. The matrix inversion $\mathbf{A}^{-1}$ can be readily performed by computers. For large matrices, however, it often is more practical to solve the equations, for example, by the Gaussian procedure of eliminating one unknown at a time.

In the application of matrices to structural analysis, loads and displacements are considered applied at the intersection of members (joints, or nodes). The loads may be resolved into moments, torques, and horizontal and vertical components. These may be assembled for each node into a vector and then all the node vectors may be combined into a force vector $\mathbf{P}$ for the whole structure.

$$\mathbf{P} = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{bmatrix} \tag{5.101}$$

Similarly, displacement corresponding to those forces may be resolved into rotations, twists, and horizontal and vertical components and assembled for the whole structure into a vector $\boldsymbol{\Delta}$.

$$\boldsymbol{\Delta} = \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_n \end{bmatrix} \tag{5.102}$$

If the structure meets requirements for application of the principle of superposition (Art. 5.10.6) and forces and displacements are arranged in the proper sequence, the vectors of forces and displacements are related by

$$\mathbf{P} = \mathbf{K}\boldsymbol{\Delta} \tag{5.103a}$$

$$\boldsymbol{\Delta} = \mathbf{FP} \tag{5.103b}$$

where $\mathbf{K}$ = stiffness matrix of the whole structure
$\mathbf{F}$ = flexibility matrix of the whole structure = $\mathbf{K}^{-1}$

The stiffness matrix $\mathbf{K}$ transform displacements into loads. The flexibility matrix $\mathbf{F}$ transforms loads into displacements. The elements of $\mathbf{K}$ and $\mathbf{F}$ are functions of material properties, such as the modules of elasticity; geometry of the structure; and sectional properties of members of the structure, such as area and moment of inertia. $\mathbf{K}$ and $\mathbf{F}$ are square matrices; that is, the number of rows in each equals the number of columns. In addition, both matrices are symmetrical; that is, in each matrix, the columns and rows may be interchanged without changing the matrix. Thus, $K_{ij} = K_{ji}$, and $F_{ij} = F_{ji}$, where i indicates the row in which an element is located and j the column.

Influence Coefficients. Elements of the stiffness and flexibility matrices are influence are coefficients. Each element is derived by computing the displacements (or forces) occurring at nodes when a unit displacement (or force) is imposed at one node, while all other displacements (or forces) are taken as zero.

Let Δ_i be the ith element of matrix $\mathbf{\Delta}$. Then a typical element F_{ij} of $\mathbf{F}$ gives the displacement of anode i in the direction of Δ_i when a unit force acts at a node j in the direction of force P_j and no other forces are acting on the structure. The jth column of $\mathbf{F}$, therefore, contains all the nodal displacements induced by a unit force acting at node j in the direction of P_j.

Similarly, Let P_i be the ith element of matrix $\mathbf{P}$. Then, a typical element K_{ij} of $\mathbf{K}$ gives the force at a node i in the direction of P_i when a node j is given a unit displacement in the direction of displacement Δ_j and no other displacements are permitted. The jth column of $\mathbf{K}$, therefore, contains all the nodal forces caused by a unit displacement of node j in the direction of Δ_j.

Application to a Beam. A general method for determining the forces and moments in a continuous beam is as follows: Remove as many redundant supports or members as necessary to make the structure statically determinant. Compute for the actual loads the deflections or rotations of the statically determinate structure in the direction of the unknown forces and couples exerted by the removed supports and members. Then, in terms of these forces and couples, treated as variables, compute the corresponding deflections or rotations the forces and couples produce in the statically determinate structure (see Arts. 5.5.16 and 5.10.4). Finally, for each redundant support or member write equations that give the known rotations or deflections of the original structure in terms of the deformations of the statically determinate structure.

For example, one method of finding the reactions of the continuous beam AC in Fig. 5.59a is to remove supports 1, 2, and 3 temporarily. The beam is now simply supported between A and C, and the reactions and moments can be computed from the laws of equilibrium. Beam AC deflects at points 1, 2, and 3, whereas we know that the continuous beam is prevented from deflecting at these points by the supports there. This information enables us to write three equations in terms of the three unknown reactions that were eliminated to make the beam statically determinate.

To determine the equations, assume that nodes exist at the location of the supports 1, 2, and 3. Then, for the actual loads, compute the vertical deflections d_1, d_2, and d_3 of simple beam AC at nodes 1, 2, and 3, respectively (Fig. 5.59b). Next, form two vectors, $\mathbf{d}$ with element d_1, d_2 and $\mathbf{R}$ with the unknown reactions R_1 at node 1, R_2 at node 2, and R_3 at node 3 as elements. Since the beam may be assumed to be linearly elastic, set $\mathbf{d} = \mathbf{FR}$, where $\mathbf{F}$ is the flexibility matrix for simple beam AC. The elements y_{ij} of $\mathbf{F}$ are influence coefficients. To determine them, calculate column 1 of $\mathbf{F}$ as the deflections y_{11}, y_{21}, and y_{31} at nodes 1, 2, and 3, respectively, when a unit force is applied at node 1 (Fig. 5.59c). Similarly, compute column 2 of $\mathbf{F}$ for a unit force at node 2 (Fig. 5.59d) and column 3 for a unit force at node 3 (Fig. 5.59e). The three equations then are given by

$$\begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \tag{5.104}$$

The solution may be represented by $\mathbf{R} = \mathbf{F}^{-1}\mathbf{d}$ and obtained by matrix or algebraic methods. See also Art. 5.13.

5.11 CONTINUOUS BEAMS AND FRAMES

Fixed-end beams, continuous beams, continuous trusses, and rigid frames are statically indeterminate. The equations of equilibrium are not sufficient for the deter-

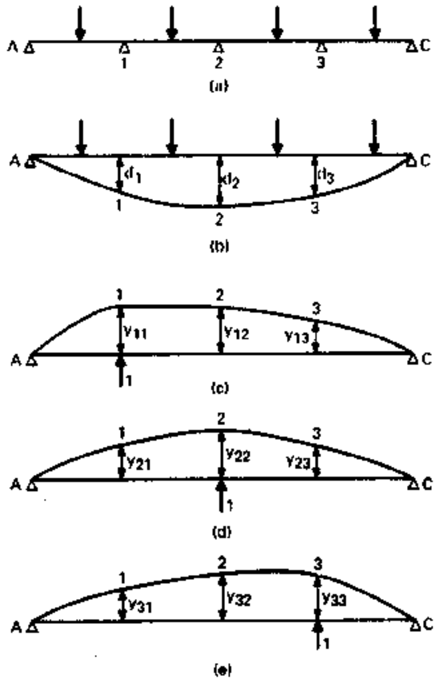


FIGURE 5.59 Determination of reactions of continuous beam *AC*: (*a*) Loaded beam with supports at points 1, 2, and 3. (*b*) Deflection of beam when supports are removed. (*c*) to (*e*) Deflections when a unit load is applied successively at points 1, 2, and 3.

mination of all the unknown forces and moments. Additional equations based on a knowledge of the deformation of the member are required.

Hence, while the bending moments in a simply supported beam are determined only by the loads and the span, bending moments in a statically indeterminate member are also a function of the geometry, cross-sectional dimensions, and modulus of elasticity.

5.11.1 Sign Convention

For computation of end moments in continuous beams and frames, the following sign convention is most convenient: A moment acting at an end of a member or at a joint is positive if it tends to rotate the joint clockwise, negative if it tends to rotate the joint counterclockwise.

Similarly, the angular rotation at the end of a member is positive if in a clockwise direction, negative if counterclockwise. Thus, a positive end moment produces a positive end rotation in a simple beam.

For ease in visualizing the shape of the elastic curve under the action of loads and end moments, bending-moment diagrams should be plotted on the tension side of each member. Hence, if an end moment is represented by a curved arrow, the arrow will point in the direction in which the moment is to be plotted.

5.11.2 Carry-Over Moments

When a member of a continuous beam or frame is loaded, bending moments are induced at the ends of the member as well as between the ends. The magnitude of the end moments depends on the magnitude and location of the loads, the geometry of the member, and the amount of restraint offered to end rotation of the member by other members connected to it. Because of the restraint, end moments are induced in the connecting members, in addition to end moments that may be induced by loads on those spans.

If the far end of a connecting member is restrained by support conditions against rotation, a resisting moment is induced at that end. That moment is called a carry-over moment. The ratio of the carry-over moment to the other end moment is called carry-over factor. It is constant for the member, independent of the magnitude and direction of the moments to be carried over. Every beam has two carry-over factors, one directed toward each end.

As pointed out in Art. 5.10.6, analysis of a continuous span can be simplified by treating it as a simple beam subjected to applied end moments. Thus, it is convenient to express the equations for carry-over factors in terms of the end rotations of simple beams: Convert a continuous member LR to a simple beam with the same span L. Apply a unit moment to one end (Fig. 5.60). The end rotation at the support where the moment is applied is α, and at the far end, the rotation is β. By the dummy-load method (Art. 5.10.4), if x is measured from the β end,

$$\alpha = \frac{1}{L^2} \int_0^L \frac{x^2}{EI_x} \, dx \tag{5.105}$$

$$\beta = \frac{1}{L^2} \int_0^L \frac{x(L - x)}{EI_x} \, dx \tag{5.106}$$

in which I_x = moment of inertia at a section a distance of x from the β end
E = modulus of elasticity

In accordance with the reciprocal theorem (Art. 5.10.5) β has the same value regardless of the beam end to which the unit moment is applied (Fig. 5.60). For prismatic beams (I_x = constant),

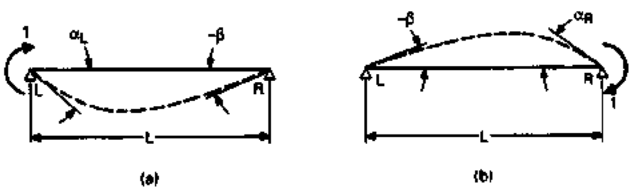


FIGURE 5.60 End rotations of a simple beam LR when a unit moment is applied (a) at end L and (b) at end R.

$$\alpha_L = \alpha_R = \frac{L}{3EI} \tag{5.107}$$

$$\beta = \frac{L}{6EI} \tag{5.108}$$

Carry-Over Factors. The preceding equations can be used to determine carry-over factors for any magnitude of end restraint. The carry-over factors toward fixed ends, however, are of special importance.

The bending-moment diagram for a continuous span *LR* that is not loaded except for a moment *M* applied at end *L* is shown in Fig. 5.61*a*. For determination of the carry-over factor C_R toward *R*, that end is assumed fixed (no rotation can occur there). The carry-over moment to *R* then is $C_R M$. The moment diagram in Fig. 5.61*a* can be resolved into two components: a simple beam with *M* applied at *L* (Fig. 5.61*b*) and a simple beam with $C_R M$ applied at *R* (Fig. 5.61*c*). As indicated in Fig. 5.61*d*, *M* causes an angle change at *R* of $-\beta$. As shown in Fig. 5.61*e*, $C_R M$ induces an angle change at *R* of $C_R M \alpha_R$. Since the net angle change at *R* is zero (Fig. 5.61*f*), $C_R M \alpha_R - M\beta = 0$, from which

$$C_R = \frac{\beta}{\alpha_R} \tag{5.109}$$

Similarly, the carry-over factor toward support *L* is given by

$$C_L = \frac{\beta}{\alpha_L} \tag{5.110}$$

Since the carry-over factors are positive, the moment carried over has the same sign as the applied moment. For prismatic beams, $\beta = L/6EI$ and $\alpha = L/3EI$. Hence,

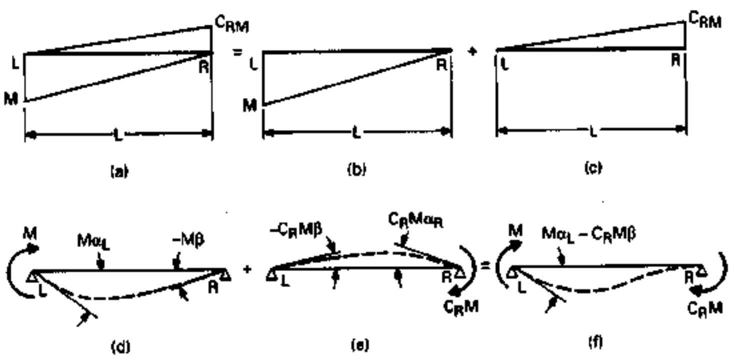


FIGURE 5.61 Effect of applying an end moment *M* to any span of a continuous beam: (*a*) An end moment $C_R M$ is induced at the opposite end. (*b*) and (*c*) The moment diagram in (*a*) is resolved into moment diagrams for a simple beam. (*d*) and (*e*) Addition of the end rotations corresponding to conditions (*b*) and (*c*) yields (*f*), the end rotations induced by *M* in the beam shown in (*a*)

$$C_L = C_R = \frac{L}{6EI}\frac{3EI}{L} = \frac{1}{2} \tag{5.111}$$

For beams with variable moment of inertia, β and α can be determined from Eqs. (5.105) and (5.106) and the carry-over factors from Eqs. (5.109) and (5.110).

If an end of a beam is free to rotate, the carry-over factor toward that end is zero.

5.11.3 Fixed-End Stiffness

The fixed-end stiffness of a beam is defined as the moment that is required to induce a unit rotation at the support where it is applied while the other end of the beam is fixed against rotation. Stiffness is important because, in the moment-distribution method, it determines the proportion of the total moment applied at a joint, or intersection of members, that is distributed to each member of the joint.

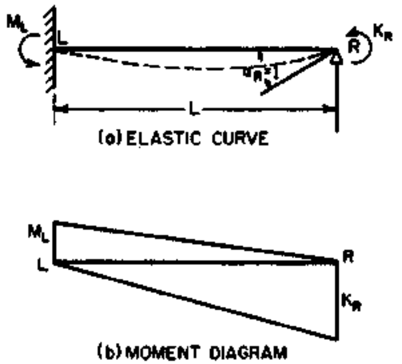


FIGURE 5.62 Determination of fixed-end stiffness: (*a*) elastic curve for moment K_R causing a unit end rotation; (*b*) the moment diagram for condition (*a*).

In Fig. 5.62*a*, the fixed-end stiffness of beam *LR* at end *R* is represented by K_R. When K_R is applied to beam *LR* at *R*, a moment $M_L = C_L K_R$ is carried over to end *L*, where C_L is the carry-over factor toward *L* (see Art. 5.11.2). K_R induces an angle change α_R at *R*, where α_R is given by Eq. (5.105). The carry-over moment induces at *R* an angle change $-C_L k_R \beta$, where β is given by Eq. (5.106). Since, by the definition of stiffness, the total angle change at *R* is unit, $K_R\alpha_R - C_L K_R \beta = 1$, from which

$$K_R = \frac{1/\alpha_R}{1 - C_R C_L} \tag{5.112}$$

when C_R is substituted for β/α_R [see Eq. (5.109)].

In a similar manner, the stiffness at *L* is found to be

$$K_L = \frac{1/\alpha_L}{1 - C_R C_L} \tag{5.113}$$

With the use of Eqs. (5.107) and (5.111), the stiffness of a beam with constant moment of inertia is given by

$$K_L = K_R = \frac{3EI/L}{1 - 1/2 \times 1/2} = \frac{4EI}{L} \tag{5.114}$$

where L = span of the beam
E = modulus of elasticity
I = moment of inertia of beam cross section

Beam with Hinge. The stiffness of one end of a beam when the other end is free to rotate can be obtained from Eqs. (5.112) or (5.113) by setting the carry-over factor toward the hinged end equal to zero. Thus, for a prismatic beam with one end hinged, the stiffness of the beam at the other end is given by

$$K = \frac{3EI}{L} \tag{5.115}$$

This equation indicates that a prismatic beam hinged at only one end has three-fourths the stiffness, or resistance to end rotation, of a beam fixed at both ends.

5.11.4 Fixed-End Moments

A beam so restrained at its ends that no rotation is produced there by the loads is called a fixed-end beam, and the end moments are called fixed-end moments. Fixed-end moments may be expressed as the product of a coefficient and *WL*, where *W* is the total load on the span *L*. The coefficient is independent of the properties of other members of the structure. Thus, any member can be isolated from the rest of the structure and its fixed-end moments computed.

Assume, for example, that the fixed-end moments for the loaded beam in Fig. 5.63*a* are to be determined. Let M_L^F be the moment at the left end *L* and M_R^F the moment at the right end *R* of the beam. Based on the condition that no rotation is permitted at either end and that the reactions at the supports are in equilibrium with the applied loads, two equations can be written for the end moments in terms of the simple-beam end rotations, θ_L at *L* and θ_R, at *R* for the specific loading.

Let K_L be the fixed-end stiffness at *L* and K_R the fixed-end stiffness at *R*, as given by Eqs. (5.112) and (5.113). Then, by resolution of the moment diagram into simple-beam components, as indicated in Fig. 5.63*f* to *h*, and application of the superposition principle (Art. 5.10.6), the fixed-end moments are found to be

$$M_L^F = -K_L(\theta_L + C_R\theta_R) \tag{5.116}$$

$$M_R^F = -K_R(\theta_R + C_L\theta_L) \tag{5.117}$$

where C_L and C_R are the carry-over factors to *L* and *R*, respectively [Eqs. (5.109) and (5.110)]. The end rotations θ_L and θ_R can be computed by a method described in Art. 5.5.15 or 5.10.4.

Prismatic Beams. The fixed-end moments for beams with constant moment of inertia can be derived from the equations given above with the use of Eqs. (5.111) and (5.114):

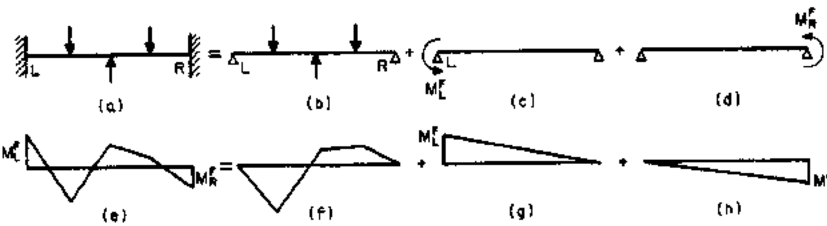


FIGURE 5.63 Determination of fixed-end moments in beam *LR*: (*a*) Loads on the fixed-end beam are resolved (*b*) to (*d*) into the sum of loads on a simple beam. (*e*) to (*h*) Bending-moment diagrams for conditions (*a*) to (*d*), respectively.

$$M_L^F = -\frac{4EI}{L}\left(\theta_L + \frac{1}{2}\theta_R\right) \tag{5.118}$$

$$M_R^F = -\frac{4EI}{L}\left(\theta_R + \frac{1}{2}\theta_L\right) \tag{5.119}$$

where L = span of the beam
E = modulus of elasticity
I = moment of inertia

For horizontal beams with gravity loads only, θ_R is negative. As a result, M_L^F is negative and M_R^F positive.

For propped beams (one end fixed, one end hinged) with variable moment of inertia, the fixed-end moments are given by

$$M_L^F = \frac{-\theta_L}{\alpha_L} \qquad \text{or} \qquad M_R^F = \frac{-\theta_R}{\alpha_R} \tag{5.120}$$

where α_L and α_R are given by Eq. (5.105). For prismatic propped beams, the fixed-end moments are

$$M_L^F = \frac{-3EI\theta_L}{L} \qquad \text{or} \qquad M_R^F = \frac{-3EI\theta_R}{L} \tag{5.121}$$

Deflection of Supports. Fixed-end moments for loaded beams when one support is displaced vertically with respect to the other support may be computed with the use of Eqs. (5.116) to (5.121) and the principle of superposition: Compute the fixed-end moments induced by the deflection of the beam when not loaded and add them to the fixed-end moments for the loaded condition with immovable supports.

The fixed-end moments for the unloaded condition can be determined directly from Eqs. (5.116) and (5.117). Consider beam LR in Fig. 5.64, with span L and support R deflected a distance d vertically below its original position. If the beam were simply supported, the angle change caused by the displacement of R would be very nearly d/L. Hence, to obtain the fixed-end moments for the deflected conditions, set $\theta_L = \theta_R = d/L$ and substitute these simple-beam end rotations in Eqs. (5.116) and (5.117):

$$M_L^F = -K_L(1 + C_R)d/L \tag{5.122}$$

$$M_R^F = -K_R(1 + C_L)d/L \tag{5.123}$$

If end L is displaced downward with respect to R, d/L would be negative and the fixed-end moments positive.

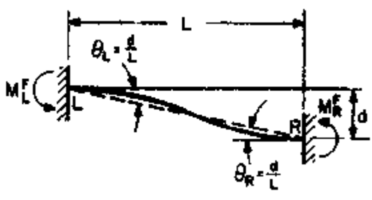


FIGURE 5.64 End moments caused by displacement d of one end of a fixed-end beam.

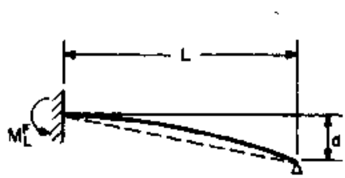


FIGURE 5.65 End moment caused by displacement d of one end of a propped beam.

For beams with constant moment of inertia, the fixed-end moments are given by

$$M_L^F = M_R^F = -\frac{6EI}{L}\frac{d}{L} \tag{5.124}$$

The fixed-end moments for a propped beam, such as beam *LR* shown in Fig. 5.65, can be obtained similarly from Eq. (5.120). For variable moment of inertia,

$$M^F = \frac{d}{L}\frac{1}{\alpha_L} \tag{5.125}$$

For a prismatic propped beam,

$$M^F = -\frac{3EI}{L}\frac{d}{L} \tag{5.126}$$

Reverse signs for downward displacement of end *L*.

Computation Aids for Prismatic Beams. Fixed-end moments for several common types of loading on beams of constant moment of inertia (prismatic beams) are given in Figs. 5.66 to 5.69. Also, the curves in Fig. 5.71 enable fixed-end moments to be computed easily for any type of loading on a prismatic beam. Before the

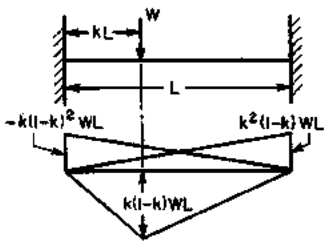


FIGURE 5.66 Moments for concentrated load on a prismatic fixed-end beam.

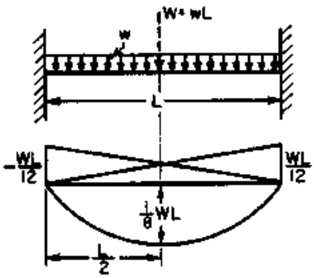


FIGURE 5.67 Moments for a uniform load on a prismatic fixed-end beam.

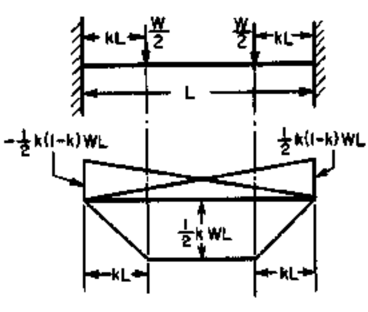


FIGURE 5.68 Moments for two equal loads on a prismatic fixed-end beam.

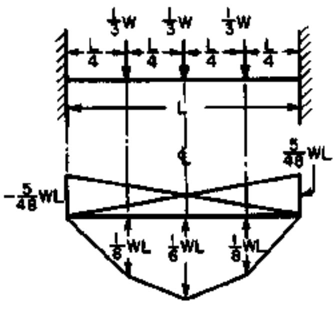


FIGURE 5.69 Moments for several equal loads on a prismatic fixed-end beam.

curves can be entered, however, certain characteristics of the loading must be calculated. These include $\bar{x}L$, the location of the center of gravity of the loading with respect to one of the loads: $G^2 = \Sigma b_n^2 P_n / W$, where $b_n L$ is the distance from each load P_n to the center of gravity of the loading (taken positive to the right); and $S^3 = \Sigma b_n^3 P_n / W$. (See Case 9, Fig. 5.70.) These values are given in Fig. 5.70 for some common types of loading.

The curves in Fig. 5.71 are entered with the location a of the center of gravity with respect to the left end of the span. At the intersection with the proper G curve, proceed horizontally to the left to the intersection within the proper S line, then vertically to the horizontal scale indicating the coefficient m by which to multiply WL to obtain the fixed-end moment. The curves solve the equations:

$$m_L = \frac{M_L^F}{WL} = G^2[1 - 3(1 - a)] + a(1 - a)^2 + S^3 \tag{5.127}$$

$$m_R = \frac{M_R^F}{WL} = G^2(1 - 3a) + a^2(1 - a) - S^3 \tag{5.128}$$

where M_L^F is the fixed-end moment at the left support and M_R^F at the right support.

As an example of the use of the curves, find the fixed-end moments in a prismatic beam of 20-ft span carrying a triangular loading of 100 kips, similar to the loading shown in Case 4, Fig. 5.70, distributed over the entire span, with the maximum intensity at the right support.

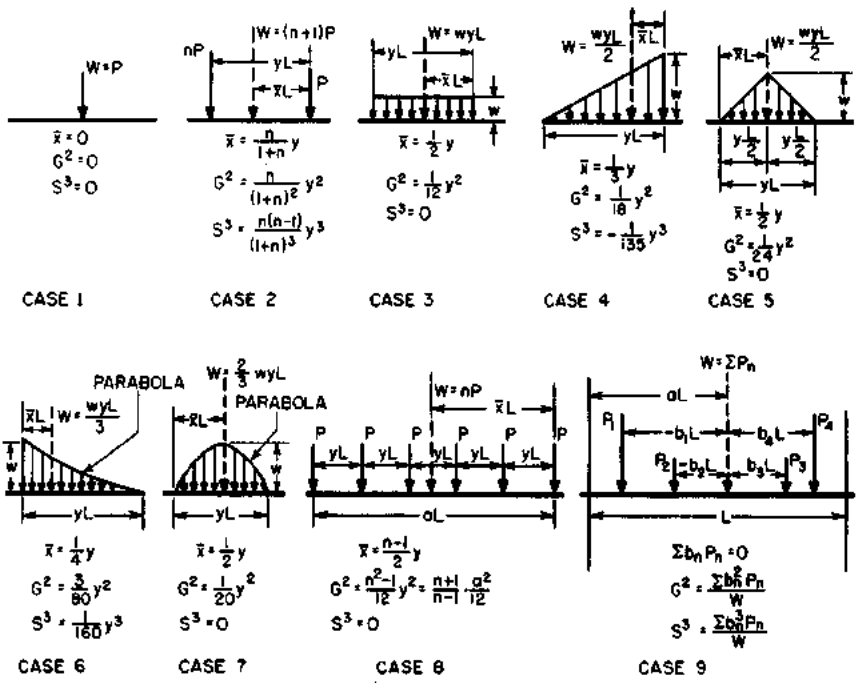


FIGURE 5.70 Characteristics of loadings.

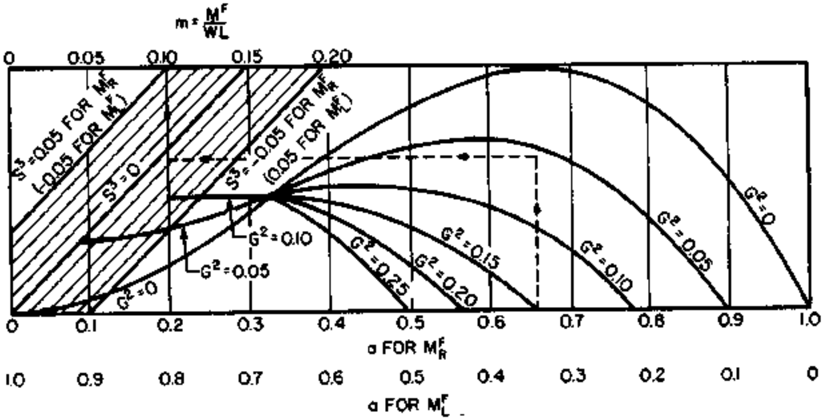


FIGURE 5.71 Chart for fixed-end moments due to any type of loading.

Case 4 gives the characteristics of the loading: $y = 1$; the center of gravity is $0.33L$ from the right support, so $a = 0.667$; $G^2 = 1/18 = 0.056$; and $S^3 = -1/135 = -0.007$. To find M_R^F, enter Fig. 5.71 with $a = 0.67$ on the upper scale at the bottom of the diagram, and proceed vertically to the estimated location of the intersection of the coordinate with the $G^2 = 0.06$ curve. Then, move horizontally to the intersection with the line for $S^3 = -0.007$, as indicated by the dash line in Fig. 5.71. Referring to the scale at the top of the diagram, find the coefficient m_R to be 0.10. Similarly, with $a = 0.67$ on the lowest scale, find the coefficient m_L to be 0.07. Hence, the fixed-end moment at the right support is $0.10 \times 100 \times 20 = 200$ ft-kips, and at the left support $-0.07 \times 100 \times 20 = -140$ ft-kips.

5.11.5 Slope-Deflection Equations

In Arts. 5.11.2 and 5.11.4, moments and displacements in a member of a continuous beam or frame are obtained by addition of their simple-beam components. Similarly, moments and displacements can be determined by superposition of fixed-end-beam components. This method, for example, can be used to derive relationships between end moments and end rotations of a beam known as slope-deflection equations. These equations can be used to compute end moments in continuous beams.

Consider a member *LR* of a continuous beam or frame (Fig. 5.72). *LR* may have a moment of inertia that varies along its length. The support *R* is displaced vertically

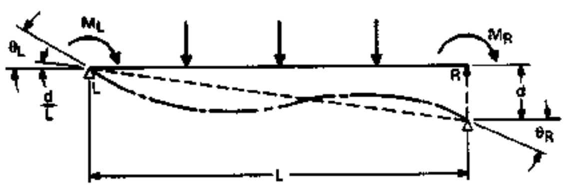


FIGURE 5.72 Elastic curve for a span *LR* of a continuous beam subjected to end moments and displacement of one end.

downward a distance d from its original position. Because of this and the loads on the member and adjacent members, LR is subjected to end moments M_L are so small that the member can be considered to rotate clockwise through an angle nearly equal to d/L, where L is the span of the beam.

Assume that rotation is prevented at ends L and R by end moments m_L at L and m_R at R. Then, by application of the principle of superposition (Art. 5.10.6) and Eqs. (5.122) and (5.123),

$$m_L = M_L^F - K_L (1 + C_R) \frac{d}{L} \tag{5.129}$$

$$m_R = M_R^F - K_R (1 + C_L) \frac{d}{L} \tag{5.130}$$

where M_L^F = fixed-end moment at L due to the load on LR
M_R^F = fixed-end moment at R due to the load on LR
K_L = fixed-end stiffness at end L
K_R = fixed-end stiffness at end R
C_L = carry-over factor toward end L
C_R = carry-over factor toward end R

Since ends L and R are not fixed but actually undergo angle changes θ_L and θ_R at L and R, respectively, the joints must now be permitted to rotate while an end moment M'_L is applied at L and an end moment M'_R at R to produce those angle changes (Fig. 5.73). With the use of the definitions of carry-over factor (Art. 5.11.2) and fixed-end stiffness (Art. 5.11.3), these moments are found to be

$$M'_L = K_L(\theta_L + C_R\theta_R) \tag{5.131}$$

$$M'_R = K_R(\theta_R + C_L\theta_L) \tag{5.132}$$

The slope-deflection equations for LR then result from addition of M'_L to m_L, which yields M_L, and of M'_R to m_R, which yields M_R:

$$M_L = K_L(\theta_L + C_R\theta_R) + M_L^F - K_L (1 + C_R) \frac{d}{L} \tag{5.133}$$

$$M_R = K_R(\theta_R + C_L\theta_L) + M_R^F - K_R(1 + C_L) \frac{d}{L} \tag{5.134}$$

For beams with constant moment of inertia, the slope-deflection equations become

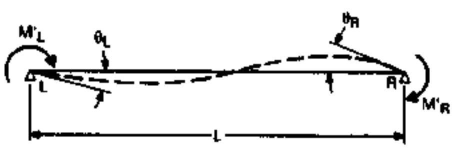


FIGURE 5.73 Elastic curve for a simple beam *LR* subjected to end moments.

$$M_L = \frac{4EI}{L}\left(\theta_L + \frac{1}{2}\theta_R\right) + M_L^F - \frac{6EI}{L}\frac{d}{L} \tag{5.135}$$

$$M_R = \frac{4EI}{L}\left(\theta_R + \frac{1}{2}\theta_L\right) + M_R^F - \frac{6EI}{L}\frac{d}{L} \tag{5.136}$$

where E = modulus of elasticity
I = moment of inertia of the cross section

Note that if end L moves downward with respect to R, the sign for d in the preceding equations is changed.

If the end moments M_L and M_R are known and the end rotations are to be determined, Eqs. (5.131) to (5.134) can be solved for θ_L and θ_R or derived by superposition of simple-beam components, as is done in Art. 5.11.4. For beams with moment of inertia varying along the span:

$$\theta_L = (M_L - M_L^F)\,\alpha_L - (M_R - M_R^F)\,\beta + \frac{d}{L} \tag{5.137}$$

$$\theta_R = (M_R - M_R^F)\,\alpha_R - (M_L - M_L^F)\,\beta + \frac{d}{L} \tag{5.138}$$

where α is given by Eq. (5.105) and β by Eq. (5.106). For beams with constant moment of inertia:

$$\theta_L = \frac{L}{3EI}(M_L - M_L^F) - \frac{L}{6EI}(M_R - M_R^F) + \frac{d}{L} \tag{5.139}$$

$$\theta_R = \frac{L}{3EI}(M_R - M_R^F) - \frac{L}{6EI}(M_L - M_L^F) + \frac{d}{L} \tag{5.140}$$

The slope-deflection equations can be used to determine end moments and rotations of the spans of continuous beams by writing compatibility and equilibrium equations for the conditions at each support. For example, the sum of the moments at each support must be zero. Also, because of continuity, the member must rotate through the same angle on both sides of every support. Hence, M_L for one span, given by Eq. (5.133) or (5.135), must be equal to $-M_R$ for the adjoining span, given by Eq. (5.134) or (5.136), and the end rotation θ at that support must be the same on both sides of the equation. One such equation with the end rotations at the supports as the unknowns can be written for each support. With the end rotations determined by simultaneous solution of the equations, the end moments can be computed from the slope-deflection equations and the continuous beam can now be treated as statically determinate.

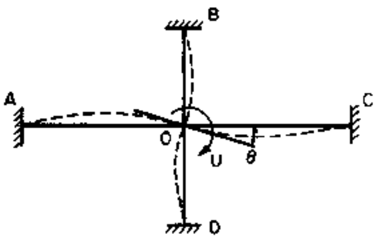
See also Arts. 5.11.9 and 5.13.2.

(C. H. Norris et al., "Elementary Structural Analysis," 4th ed., McGraw-Hill Book Company, New York.)

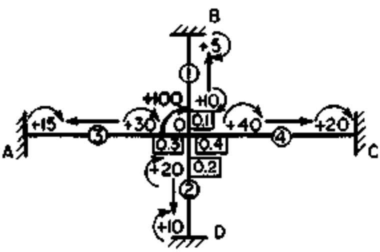
5.11.6 Moment Distribution

The frame in Fig. 5.74 consists of four prismatic members rigidly connected together at O at fixed at ends A, B, C, and D. If an external moment U is applied at

O, the sum of the end moments in each member at O must be equal to U. Furthermore, all members must rotate at O through the same angle θ, since they are assumed to be rigidly connected there. Hence, by the definition of fixed-end stiffness, the proportion of U induced in the end of each member at O is equal to the ratio of the stiffness of that member to the sum of the stiffnesses of all the members at the joint (Art. 5.11.3).



(a) ELASTIC CURVE FOR UNBALANCED MOMENT AT JOINT O



(b) STIFFNESSES AND DISTRIBUTION FACTORS FOR A FRAME

FIGURE 5.74 Effect of an unbalanced moment at a joint in a frame.

Suppose a moment of 100 ft-kips is applied at O, as indicated in Fig. 5.74*b*. The relative stiffness (or I/L) is assumed as shown in the circle on each member. The distribution factors for the moment at O are computed from the stiffnesses and shown in the boxes. For example, the distribution factor for OA equals its stiffness divided by the sum of the stiffnesses of all the members at the joint: $3/(3 + 2 + 4 + 1) = 0.3$. Hence, the moment induced in OA at O is $0.3 \times 100 = 30$ ft-kips. Similarly, OB gets 10 ft-kips, OC 40 ft-kips and OD 20 ft-kips.

Because the far ends of these members are fixed, one-half of these moments are carried over to them (Art. 5.11.2). Thus $M_{AO} = 0.5 \times 30 = 15$; $M_{BO} = 0.5 \times 10 = 5$; $M_{CO} = 0.5 \times 40 = 20$; and $M_{DO} = 0.5 \times 20 = 10$.

Most structures consist of frames similar to the one in Fig. 5.74, or even simpler, joined together. Though the ends of the members are not fixed, the technique employed for the frame in Fig. 5.74*b* can be applied to find end moments in such continuous structures.

Before the general method is presented, one short cut is worth noting. Advantage can be taken when a member has a hinged end to reduce the work of distributing moments. This is done by using the true stiffness of a member instead of the fixed-end stiffness. (For a prismatic beam with one end hinged, the stiffness is three-fourth the fixed-end stiffness; for a beam with variable I, it is equal to the fixed-end stiffness times $1 - C_L C_R$, where C_L and C_R are the carry-over factors for the beam.) Naturally, the carry-over factor toward the hinge is zero.

When a joint is neither fixed nor pinned but is restrained by elastic members connected there, moments can be distributed by a series of converging approximations. All joints are locked against rotation. As a result, the loads will create fixed-end moments at the ends of every member. At each joint, a moment equal to the algebraic sum of the fixed-end moments there is required to hold it fixed. Then, one joint is unlocked at a time by applying a moment equal but opposite in sign to the moment that was needed to prevent rotation. The unlocking moment must be distributed to the members at the joint in proportion to their fixed-end stiffnesses and the distributed moments carried over to the far ends.

After all joints have been released at least once, it generally will be necessary to repeat the process—sometimes several times—before the corrections to the fixed-end moments become negligible. To reduce the number of cycles, the unlocking of joints should start with those having the greatest unbalanced moments.

Suppose the end moments are to be found for the prismatic continuous beam *ABCD* in Fig. 5.75. The *I*/*L* values for all spans are equal; therefore, the relative fixed-end stiffness for all members is unity. However, since *A* is a hinged end, the computation can be shortened by using the actual relative stiffness, which is ¾. Relative stiffnesses for all members are shown in the circle on each member. The distribution factors are shown in boxes at each joint.

The computation starts with determination of fixed-end moments for each member (Art. 5.11.4). These are assumed to have been found and are given on the first line in Fig. 5.75. The greatest unbalanced moment is found from inspection to be at hinged end *A*; so this joint is unlocked first. Since there are no other members at the joint, the full unlocking moment of +400 is distributed to *AB* at *A* and one-half of this is carried over to *B*. The unbalance at *B* now is +400 − 480 plus the carry-over of +200 from *A*, or a total of +120. Hence, a moment of −120 must be applied and distributed to the members at *B* by multiplying by the distribution factors in the corresponding boxes.

The net moment at *B* could be found now by adding the entries for each member at the joint. However, it generally is more convenient to delay the summation until the last cycle of distribution has been completed.

The moment distributed to *BA* need not be carried over to *A*, because the carry-over factor toward the hinged end is zero. However, half the moment distributed to *BC* is carried over to *C*.

Similarly, joint *C* is unlocked and half the distributed moments carried over to *B* and *D*, respectively. Joint *D* should not be unlocked, since it actually is a fixed end. Thus, the first cycle of moment distribution has been completed.

The second cycle is carried out in the same manner. Joint *B* is released, and the distributed moment in *BC* is carried over to *C*. Finally, *C* is unlocked, to complete the cycle. Adding the entries for the end of each member yields the final moments.

5.11.7 Maximum Moments in Continuous Frames

In design of continuous frames, one objective is to find the maximum end moments and interior moments produced by the worst combination of loading. For maximum moment at the end of a beam, live load should be placed on that beam and on the

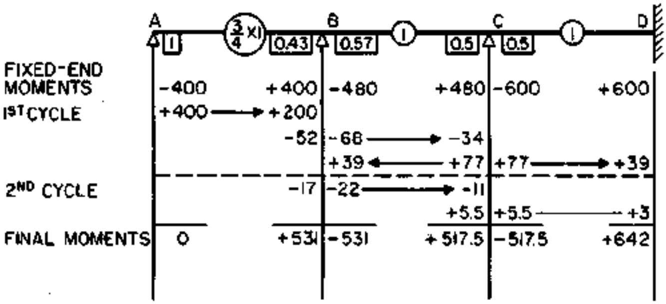


FIGURE 5.75 Moment distribution by converging approximations for a continuous beam.

beam adjoining the end for which the moment is to be computed. Spans adjoining these two should be assumed to be carrying only dead load.

For maximum midspan moments, the beam under consideration should be fully loaded, but adjoining spans should be assumed to be carrying only dead load.

The work involved in distributing moments due to dead and live loads in continuous frames in buildings can be greatly simplified by isolating each floor. The tops of the upper columns and the bottoms of the lower columns can be assumed fixed. Furthermore, the computations can be condensed considerably by following the procedure recommended in "Continuity in Concrete Building Frames." EB033D, Portland Cement Association, Skokie, IL 60077, and indicated in Fig. 5.74.

Figure 5.74 presents the complete calculation for maximum end and midspan moments in four floor beams *AB*, *BC*, *CD*, and *DE*. Building columns are assumed to be fixed at the story above and below. None of the beam or column sections is known to begin with; so as a start, all members will be assumed to have a fixed-end stiffness of unity, as indicated on the first line of the calculation.

On the second line, the distribution factors for each end of the beams are shown, calculated from the stiffnesses (Arts. 5.11.3 and 5.11.4). Column stiffnesses are not shown, because column moments will not be computed until moment distribution to the beams has been completed. Then the sum of the column moments at each joint may be easily computed, since they are the moments needed to make the sum of the end moments at the joint equal to zero. The sum of the column moments at each joint can then be distributed to each column there in proportion to its stiffness. In this example, each column will get one-half the sum of the column moments.

Fixed-end moments at each beam end for dead load are shown on the third line, just above the heavy line, and fixed-end moments for live plus dead load on the fourth line. Corresponding midspan moments for the fixed-end condition also are shown on the fourth line and, like the end moments, will be corrected to yield actual midspan moments.

For maximum end moment at *A*, beam *AB* must be fully loaded, but *BC* should carry dead load only. Holding *A* fixed, we first unlock joint *B*, which has a total-load fixed-end moment of +172 in *BA* and a dead-load fixed-end moment of −37 in *BC*. The releasing moment required, therefore, is −(172 − 37), or − 135. When *B* is released, a moment of −135 × ¼ is distributed to *BA* One-half of this is carried over to *A*, or −135 × ¼ × ½ = −17. This value is entered as the carry-over at *A* on the fifth line in Fig. 5.76. Joint *B* is then relocked.

	A			B			C			D		E
1. RELATIVE STIFFNESS		$K^F=1$			$K^F=1$			$K^F=1$			$K^F=1$	
2. DISTRIBUTION FACTOR	1/3		1/4	1/4		1/4	1/4		1/4	1/4		1/3
3. F.E.M. DEAD LOAD	—		+91	−37		+37	−70		+70	−59		—
4. F.E.M. TOTAL LOAD	−172	+99	+172	−78	+73	+78	−147	+85	+147	−126	+63	+126
5. CARRY-OVER	−17	+11	+29	−1	+1	−2	−11	+7	+14	−21	+13	+7
6. ADDITION	−189	+18	+201	−79	−1	+76	−158	+9	+161	−147	+5	+133
7. DISTRIBUTION	+63		−30	−30		+21	+21		−4	−4		−44
8. MAX. MOMENTS	−126	+128	+171	−109	+73	+97	−137	+101	+157	−151	+81	+89

FIGURE 5.76 Bending moments in a continuous frame obtained by moment distribution.

At A, for which we are computing the maximum moment, we have a total-load fixed-end moment of -172 and a carry-over of -17, making the total -189, shown on the sixth line. To release A, a moment of $+189$ must be applied to the joint. Of this, $189 \times \frac{1}{3}$, or 63, is distributed to AB, as indicated on the seventh line of the calculation. Finally, the maximum moment at A is found by adding lines 6 and 7: $-189 + 63 = -126$.

For maximum moment at B, both AB and BC must be fully loaded but CD should carry only dead load. We begin the determination of the moment at B by first releasing joints A and C, for which the corresponding carry-over moments at BA and BC are $+29$ and $-(+78 - 70) \times \frac{1}{4} \times \frac{1}{2} = -1$, shown on the fifth line in Fig. 5.76. These bring the total fixed-end moments in BA and BC to $+201$ and -79, respectively. The releasing moment required is $-(201 - 79) = -122$. Multiplying this by the distribution factors for BA and BC when joint B is released, we find the distributed moments, -30, entered on line 7. The maximum end moments finally are obtained by adding lines 6 and 7: $+171$ at BA and -109 at BC. Maximum moments at C, D, and E are computed and entered in Fig. 5.76 in a similar manner. This procedure is equivalent to two cycles of moment distribution.

The computation of maximum midspan moments in Fig. 5.76 is based on the assumption that in each beam the midspan moment is the sum of the simple-beam midspan moment and one-half the algebraic difference of the final end moments (the span carries full load but adjacent spans only dead load). Instead of starting with the simple-beam moment, however, we begin with the midspan moment for the fixed-end condition and apply two corrections. In each span, these corrections are equal to the carry-over moments entered on line 5 for the two ends of the beam multiplied by a factor.

For beams with variable moment of inertia, the factor is $+\frac{1}{2}[(1/C^F) + D - 1]$ where C^F is the fixed-end-carry-over factor toward the end for which the correction factor is being computed and D is the distribution factor for that end. The plus sign is used for correcting the carry-over at the right end of a beam, and the minus sign for the carry-over at the left end. For prismatic beams, the correction factor becomes $\pm\frac{1}{2}(1 + D)$.

For example, to find the corrections to the midspan moment in AB, we first multiply the carry-over at A on line 5, -17, by $-\frac{1}{2}(1 + \frac{1}{3})$. The correction, $+11$, is also entered on the fifth line. Then, we multiply the carry-over at B, $+ 29$, by $+\frac{1}{2}(1 + \frac{1}{4})$ and enter the correction, $+18$, on line 6. The final midspan moment is the sum of lines 4, 5, and 6: $+99 + 11 + 18 = +128$. Other midspan moments in Fig. 5.74 are obtained in a similar manner.

See also Arts. 5.11.9 and 5.11.10.

5.11.8 Moment-Influence Factors

In certain types of framing, particularly those in which different types of loading conditions must be investigated, it may be convenient to find maximum end moments from a table of moment-influence factors. This table is made up by listing for the end of each member in the structure the moment induced in that end when a moment (for convenience, $+1000$) is applied to every joint successively. Once this table has been prepared, no additional moment distribution is necessary for computing the end moments due to any loading condition.

For a specific loading pattern, the moment at any beam end M_{AB} may be obtained from the moment-influence table by multiplying the entries under AB for the various joints by the actual unbalanced moments at those joints divided by 1000, and summing (see also Art. 5.11.9 and Table 5.6).

5.11.9 Procedure for Sidesway

Computations of moments due to sidesway, or drift, in rigid frames is conveniently executed by the following method:

1. Apply forces to the structure to prevent sidesway while the fixed-end moments due to loads are distributed.
2. Compute the moments due to these forces.
3. Combine the moments obtained in Steps 1 and 2 to eliminate the effect of the forces that prevented sidesway.

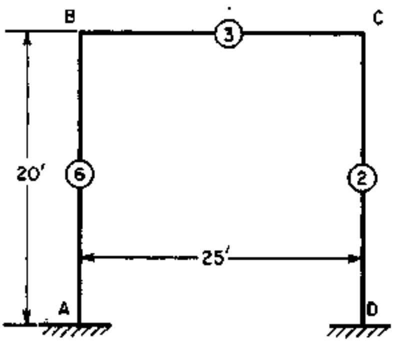


FIGURE 5.77 Rigid frame.

Suppose the rigid frame in Fig. 5.77 is subjected to a 2000-lb horizontal load acting to the right at the level of beam *BC*. The first step is to compute the moment-influence factors (Table 5.6) by applying moments of $+1000$ at joints *B* and *C*, assuming sidesway prevented.

Since there are no intermediate loads on the beams and columns, the only fixed-end moments that need be considered are those in the columns resulting from lateral deflection of the frame caused by the horizontal load. This deflection, however is not known initially. So assume an arbitrary deflection, which produces a fixed-end moment of $-1000M$ at the top of column *CD*. *M* is an unknown constant to be determined from the fact that the sum of the shears in the deflected columns must be equal to the 2000-lb load. The same deflection also produces a moment of $-1000M$ at the bottom of *CD* [see Eq. (5.126)].

From the geometry of the structure, furthermore, note that the deflection of *B* relative to *A* is equal to the deflection of *C* relative to *D*. Then, according to Eq. (5.126) the fixed-end moments in the columns are proportional to the stiffnesses of

TABLE 5.6 Moment-Influence Factors for Fig. 5.77

Member	$+1000$ at *B*	$+1000$ at *C*
AB	351	-105
BA	702	-210
BC	298	210
CB	70	579
CD	-70	421
DC	-35	210

the columns and hence are equal in AB to $-1000M \times \frac{6}{2} = -3000M$. The column fixed-end moments are entered in the first line of Table 5.7, which is called a moment-collection table.

In the deflected position of the frame, joints B and C are unlocked. First, apply a releasing moment of $+3000M$ at B and distribute it by multiplying by 3 the entries in the column marked "+1000 at B" in Table 5.6. Similarly, a releasing moment of $+1000M$ is applied at C and distributed with the aid of Table 5.6. The distributed moments are entered in the second and third lines of Table 5.7. The final moments are the sum of the fixed-end moments and the distributed moments and are given in the fifth line.

Isolating each column and taking moments about one end, we find that the overturning moment due to the shear is equal to the sum of the end moments. There is one such equation for each column. Addition of these equations, noting that the sum of the shears equals 2000 lb, yields

$$-M(2052 + 1104 + 789 + 895) = -2000 \times 20$$

from which $M = 8.26$. This value is substituted in the sidesway totals in Table 5.7 to yield the end moments for the 2000-lb horizontal load.

Suppose now a vertical load of 4000 lb is applied to BC of the rigid frame in Fig. 5.77, 5 ft from B. Tables 5.6 and 5.7 can again be used to determine the end moments with a minimum of labor:

The fixed-end moment at B, with sidesway prevented, is $-12{,}800$, and at C $+3200$. With the joints locked, the frame is permitted to move laterally an arbitrary amount, so that in addition to the fixed-end moments due to the 4000-lb load, column fixed-end moments of $-3000M$ at B and $-1000M$ at C are induced. Table 5.7 already indicates the effect of relieving these column moments by unlocking joints B and C. We now have to superimpose the effect of releasing joints B and C to relieve the fixed-end moments for the vertical load. This we can do with the aid of Table 5.6. The distribution is shown in the lower part of Table 5.7. The sums of the fixed-end moments and distributed moments for the 4000-lb load are shown on the line "No-sidesway sum."

The unknown M can be evaluated from the fact that the sum of the horizontal forces acting on the columns must be zero. This is equivalent to requiring that the sum of the column end moments equals zero:

$$-M(2052 + 1104 + 789 + 895) + 4826 + 9652 - 2244 - 1120 = 0$$

from which $M = 2.30$. This value is substituted in the sidesway total in Table 5.7 to yield the sidesway moments for the 4000-lb load. The addition of these moments to the totals for no sidesway yields the final moments.

This procedure enables one-story bents with straight beams to be analyzed with the necessity of solving only one equation with one unknown regardless of the number of bays. If the frame is several stories high, the procedure can be applied to each story. Since an arbitrary horizontal deflection is introduced at each floor or roof level, there are as many unknowns and equations as there are stories.

The procedure is more difficult to apply to bents with curved or polygonal members between the columns. The effect of the change in the horizontal projection of the curved or polygonal portion of the bent must be included in the calculations. In many cases, it may be easier to analyze the bent as a curved beam (arch).

(A. Kleinlogel, "Rigid Frame Formulas," Frederick Ungar Publishing Co., New York.)

TABLE 5.7 Moment-Collection Table for Fig. 5.77

Remarks	*AB* +	*AB* −	*BA* +	*BA* −	*BC* +	*BC* −	*CB* +	*CB* −	*CD* +	*CD* −	*DC* +	*DC* −
Sidesway, *FEM*		3,000*M*		3,000*M*						1,000*M*		1,000*M*
B moments	1,053*M*		2,106*M*		894*M*		210*M*			210*M*		105*M*
C moments		105*M*		210*M*	210*M*		579*M*		421*M*		210*M*	
Partial sum	1,053*M*	3,105*M*	2,106*M*	3,210*M*	1,104*M*		789*M*		421*M*	1,210*M*	210*M*	1,105*M*
Totals		2,052*M*		1,104*M*	1,104*M*		789*M*			789*M*		895*M*
For 2000-lb load		17,000		9,100	9,100		6,500			6,500		7,400
4000-lb load, *FEM*						12,800	3,200					
B moments	4,490		8,980		3,820		897			897		448
C moments	336		672			672		1,853		1,347		672
Partial sum	4,826		9,652		3,820	13,472	4,097	1,853		2,244		1,120
No-sidesway sum	4,826		9,652			9,652	2,244			2,244		1,120
Sidesway *M*		4,710		2,540	2,540		1,810			1,810		2,060
Totals	120		7,110			7,110	4,050			4,050		3,180

5.11.10 Rapid Approximate Analysis of Multistory Frames

Exact analysis of multistory rigid frames subjected to lateral forces, such as those from wind or earthquakes, involves lengthy calculations, and they are time-consuming and expensive, even when performed with computers. Hence, approximate methods of analysis are an alternative, at least for preliminary designs and, for some structures, for final designs.

It is noteworthy that for some buildings even the "exact" methods, such as those described in Arts. 5.11.8 and 5.11.9, are not exact. Usually, static horizontal loads are assumed for design purposes, but actually the forces exerted by wind and earthquakes are dynamic. In addition, these forces generally are uncertain in intensity, direction, and duration. Earthquake forces, usually assumed as a percentage of the mass of the building above each level, act at the base of the structure, not at each floor level as is assumed in design, and accelerations at each level vary nearly linearly with distance above the base. Also, at the beginning of a design, the sizes of the members are not known. Consequently, the exact resistance to lateral deformation cannot be calculated. Furthermore, floors, walls, and partitions help resist the lateral forces in a very uncertain way. See Art. 5.12 for a method of calculating the distribution of loads to rigid-frame bents.

Portal Method. Since an exact analysis is impossible, most designers prefer a wind-analysis method based on reasonable assumptions and requiring a minimum of calculations. One such method is the so-called "portal method."

It is based on the assumptions that points of inflection (zero bending moment) occur at the midpoints of all members and that exterior columns take half as much shear as do interior columns. These assumptions enable all moments and shears throughout the building frame to be computed by the laws of equilibrium.

Consider, for example, the roof level (Fig. 5.78*a*) of a tall building. A wind load of 600 lb is assumed to act along the top line of girders. To apply the portal method, we cut the building along a section through the inflection points of the top-story columns, which are assumed to be at the column midpoints, 6 ft down from the top of the building. We need now consider only the portion of the structure above this section.

Since the exterior columns take only half as much shear as do the interior columns, they each receive 100 lb, and the two interior columns, 200 lb. The moments at the tops of the columns equal these shears times the distance to the inflection point. The wall end of the end girder carries a moment equal to the moment in the column. (At the floor level below, as indicated in Fig. 5.78*b*, that end of the end girder carries a moment equal to the sum of the column moments.) Since the inflection point is at the midpoint of the girder, the moment at the inner end of the girder must the same as at the outer end. The moment in the adjoining girder can be found by subtracting this moment from the column moment, because the sum of the moments at the joint must be zero. (At the floor level below, as shown in Fig. 5.78*b*, the moment in the interior girder is found by subtracting the moment in the exterior girder from the sum of the column moments.)

Girder shears then can be computed by dividing girder moments by the half span. When these shears have been found, column loads can be easily computed from the fact that the sum of the vertical loads must be zero, by taking a section around each joint through column and girder inflection points. As a check, it should be noted that the column loads produce a moment that must be equal to the moments of the wind loads above the section for which the column loads were computed. For the roof level (Fig. 5.78*a*), for example, $-50 \times 24 + 100 \times 48 = 600 \times 6$.

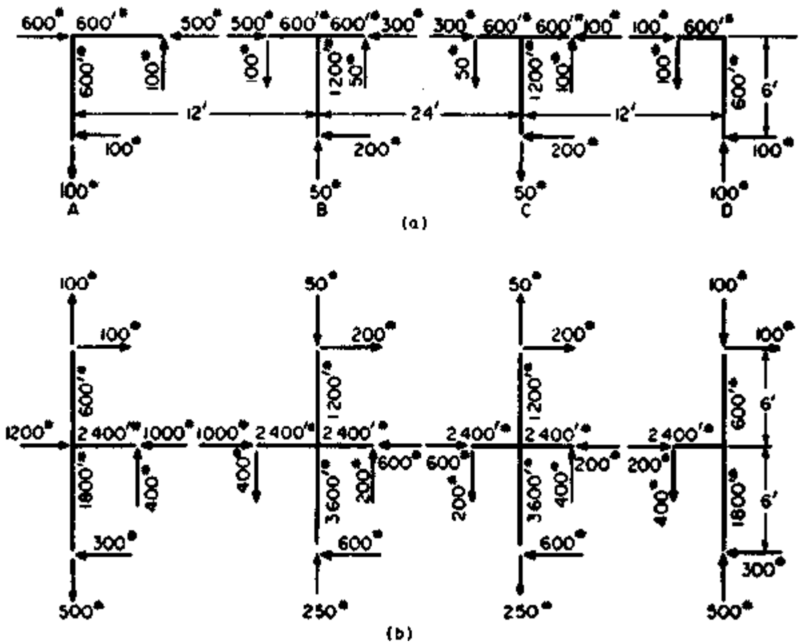


FIGURE 5.78 Portal method for computing wind stresses in a tall building.

Cantilever Method. Another wind-analysis procedure that is sometimes employed is the cantilever method. Basic assumptions here are that inflection points are at the midpoints of all members and that direct stresses in the columns vary as the distances of the columns from the center of gravity of the bent. The assumptions are sufficient to enable shears and moments in the frame to be determined from the laws of equilibrium.

For multistory buildings with height-to-width ratio of 4 or more, the Spurr modification is recommended ("Welded Tier Buildings," U.S. Steel Corp.). In this method, the moments of inertia of the girders at each level are made proportional to the girder shears.

The results obtained from the cantilever method generally will be different from those obtained by the portal method. In general, neither solution is correct, but the answers provide a reasonable estimate of the resistance to be provided against lateral deformation. (See also *Transactions of the ASCE,* Vol. 105, pp. 1713–1739, 1940.)

5.11.11 Beams Stressed into the Plastic Range

When an elastic material, such as structural steel, is loaded in tension with a gradually increasing load, stresses are proportional to strains up to the proportional limit (near the yield point). If the material, like steel, also is ductile, then it continues to carry load beyond the yield point, though strains increase rapidly with little increase in load (Fig. 5.79*a*).

Similarly, a beam made of a ductile material continues to carry more load after the stresses in the outer surfaces reach the yield point. However, the stresses will no longer vary with distance from the neutral axis, so the flexure formula [Eq. (5.54)] no longer holds. However, if simplifying assumptions are made, approximating the stress-strain relationship beyond the elastic limit, the load-carrying capacity of the beam can be computed with satisfactory accuracy.

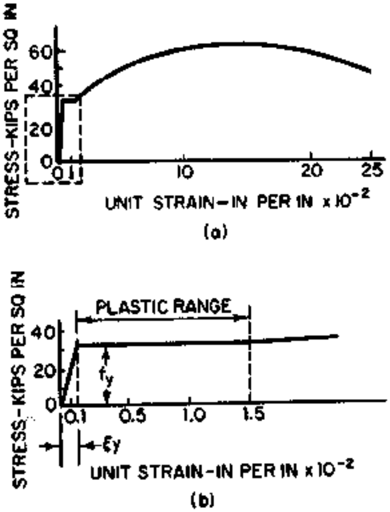


FIGURE 5.79 Stress-strain relationship for a ductile material generally is similar to the curve shown in (*a*). To simplify plastic analysis, the portion of (*a*) enclosed by the dash lines is approximated by the curve in (*b*), which extends to the range where strain hardening begins.

Modulus of rupture is defined as the stress computed from the flexure formula for the maximum bending moment a beam sustains at failure. This is not a true stress but it is sometimes used to compare the strength of beams.

For a ductile material, the idealized stress-strain relationship in Fig. 5.79*b* may be assumed. Stress is proportional to strain until the yield-point stress f_y is reached, after which strain increases at a constant stress.

For a beam of this material, the following assumptions will also be made:

1. Plane sections remain plane, strains thus being proportional to distance from the neutral axis.
2. Properties of the material in tension are the same as those in compression.
3. Its fibers behave the same in flexure as in tension.
4. Deformations remain small.

Strain distribution across the cross section of a rectangular beam, based on these assumptions, is shown in Fig. 5.80*a*. At the yield point, the unit strain is ϵ_y and the curvature ϕ_y, as indicated in (1). In (2), the strain has increased several times, but the section still remains plane. Finally, at failure, (3), the strains are very large and nearly constant across upper and lower halves of the section.

Corresponding stress distributions are shown in Fig. 5.80*b*. At the yield point, (1), stresses vary linearly and the maximum if f_y. With increase in load, more and more fibers reach the yield point, and the stress distribution becomes nearly constant, as indicated in (2). Finally, at failure, (3), the stresses are constant across the top and bottom parts of the section and equal to the yield-point stress.

The resisting moment at failure for a rectangular beam can be computed from the stress diagram for stage 3. If b is the width of the member and d its depth, then the ultimate moment for a rectangular beam is

$$M_p = \frac{bd^2}{4} f_y \qquad (5.141)$$

Since the resisting moment at stage 1 is $M_y = f_y bd^2/6$, the beam carries 50% more moment before failure than when the yield-point stress is first reached at the outer surfaces.

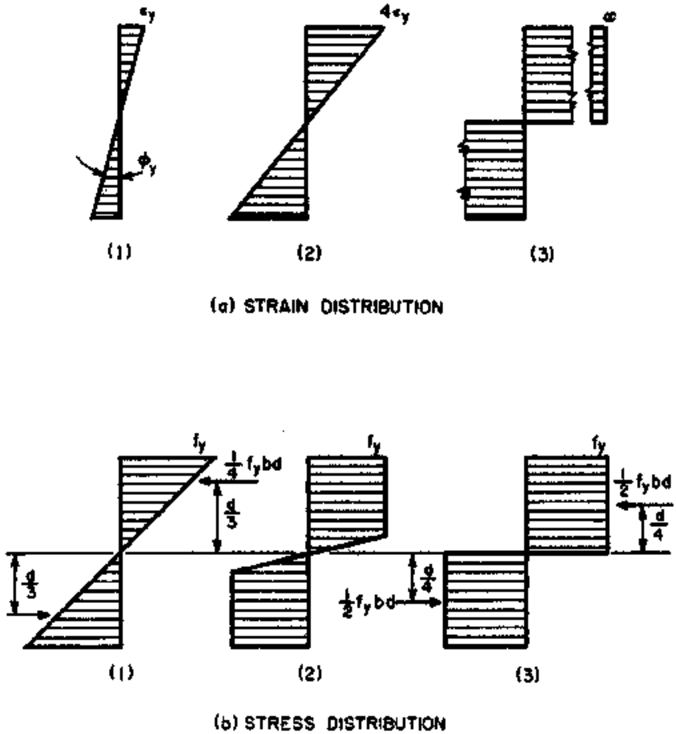


FIGURE 5.80 Strain distribution is shown in (*a*) and stress distribution in (*b*) for a cross section of a beam as it is loaded beyond the yield point, for the idealized stress-strain relationship in Fig. 5.79*b*: stage (1) shows the condition at the yield point of the outer surface; (2) after yielding starts; (3) at ultimate load.

A circular section has an M_p/M_y ratio of about 1.7, while a diamond section has a ratio of 2. The average wide-flange rolled-steel beam has a ratio of about 1.14.

Plastic Hinges. The relationship between moment and curvature in a beam can be assumed to be similar to the stress-strain relationship in Fig. 5.80*b*. Curvature ϕ varies linearly with moment until $M_y = M_p$ is reached, after which ϕ increases indefinitely at constant moment. That is, a plastic hinge forms.

Moment Redistribution. This ability of a ductile beam to form plastic hinges enables a fixed-end or continuous beam to carry more load after M_P occurs at a section, because a redistribution of moments takes place. Consider, for example, a uniformly loaded, fixed-end, prismatic beam. In the elastic range, the end moments of $M_L = M_R = WL/12$, while the midspan moment M_C is $WL/24$. The load when the yield point is reached at the outer surfaces at the beam ends is $W_y = 12M_y/L$. Under this load the moment capacity of the ends of the beam is nearly exhausted; plastic hinges form there when the moment equals M_P. As load is increased, the ends then rotate under constant moment and the beam deflects like a simply supported beam. The moment at midspan increases until the moment capacity at that section is exhausted and a plastic hinge forms. The load causing that condition is the ultimate load W_u since, with three hinges in the span, a link mechanism is formed and the member continues to deform at constant load. At the time the third hinge is formed, the moments at ends and center are all equal to M_P. Therefore, for equilibrium, $2M_P = W_uL/8$, from which $W_u = 16M_P/L$. Since for the idealized moment-curvature relationship, M_P was assumed equal to M_y, the carrying capacity due to redistribution of moments is 33% greater than W_y.

5.12 LOAD DISTRIBUTION TO BENTS AND SHEAR WALLS

Buildings must be designed to resist horizontal forces as well as vertical loads. In tall buildings, the lateral forces must be given particular attention, because if they are not properly provided for, they can collapse the structure (Art. 3.2.3). The usual procedure for preventing such disasters is to provide structural framing capable of transmitting the horizontal forces from points of application to the building foundations.

Because the horizontal loads may come from any direction, they generally are resolved into perpendicular components, and correspondingly the lateral-force-resisting framing is also placed in perpendicular directions. The maximum magnitude of load is assumed to act in each of those directions. Bents or shear walls, which act as vertical cantilevers and generally are often also used to support some of the building's gravity loads, usually are spaced at appropriate intervals for transmitting the loads to the foundations.

A bent consists of vertical trusses or continuous rigid frames located in a plane. The trusses usually are an assemblage of columns, horizontal girders, and diagonal bracing (Art. 3.2.4). The rigid frames are composed of girders and columns, with so-called **wind connections** between them to establish continuity. Shear walls are thin cantilevers braced by floors and roofs (Art. 3.2.4).

5.12.1 Diaphragms

Horizontal distribution of lateral forces to bents and shear walls is achieved by the floor and roof systems acting as diaphragms (Fig. 5.81).

To qualify as a diaphragm, a floor or roof system must be able to transmit the lateral forces to bents and shear walls without exceeding a horizontal deflection that would cause distress to any vertical element. The successful action of a diaphragm also requires that it be properly tied into the supporting framing. Designers should ensure this action by appropriate detailing at the juncture between horizontal and vertical structural elements of the building.

Diaphragms may be considered analogous to horizontal (or inclined, in the case of some roofs) plate girders. The roof or floor slab constitutes the web; the joists, beams, and girders function as stiffeners; and the bents and shear walls act as flanges.

Diaphragms may be constructed of structural materials, such as concrete, wood, or metal in various forms. Combinations of such materials are also possible. Where a diaphragm is made up of units, such as plywood, precast-concrete planks, or steel

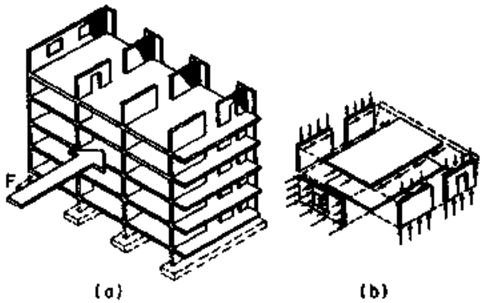


FIGURE 5.81 Floors of building distribute horizontal loads to shear walls (diaphragm action).

decking, its characteristics are, to a large degree, dependent on the attachments of one unit to another and to the supporting members. Such attachments must resist shearing stresses due to internal translational and rotational actions.

The stiffness of a horizontal diaphragm affects the distribution of the lateral forces to the bents and shear walls. For the purpose of analysis, diaphragms may be classified into three groups—rigid, semirigid or semiflexible, and flexible—although no diaphragm is actually infinitely rigid or infinitely flexible.

A **rigid diaphragm** is assumed to distribute horizontal forces to the vertical resisting elements in proportion to the relative rigidities of these elements (Fig. 5.82).

Semirigid or semiflexible diaphragms are diaphragms that deflect significantly under load, but have sufficient stiffness to distribute a portion of the load to the vertical elements in proportion to the rigidities of these elements. The action is analogous to a continuous beam of appreciable stiffness on yielding supports (Fig. 5.83). Diaphragm reactions are dependent on the relative stiffnesses of diaphragm and vertical resisting elements.

A **flexible diaphragm** is analogous to a continuous beam or series of simple beams spanning between nondeflecting supports. Thus, a flexible diaphragm is con-

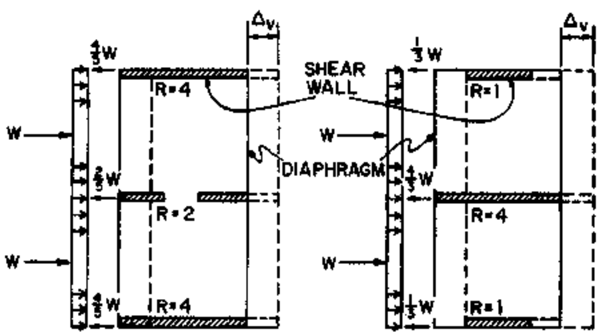


FIGURE 5.82 Horizontal section through shear walls connected by a rigid diaphragm. R = relative rigidity and Δ_v = shear-wall deflection.

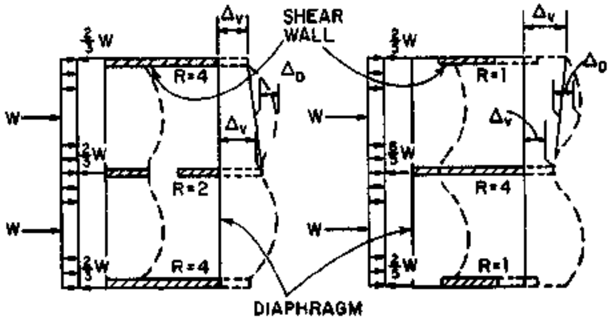


FIGURE 5.83 Horizontal sections through shear walls connected by a semirigid diaphragm. Δ_D = diaphragm horizontal deflection.

sidered to distribute the lateral forces to the vertical resisting elements in proportion to the exterior-wall tributary areas (Fig. 5.84).

A rigorous analysis of lateral-load distribution to shear walls or bents is sometimes very time-consuming, and frequently unjustified by the results. Therefore, in many cases, a design based on reasonable limits may be used. For example, the load may be distributed by first considering the diaphragm rigid, and then by considering it flexible. If the difference in results is not great, the shear walls can then be safely designed for the maximum applied load. (See also Art. 5.12.2.)

5.12.2 Torque Distribution to Shear Walls

When the line of action of the resultant of lateral forces acting on a building does not pass through the center of rigidity of a vertical, lateral-force-resisting system, distribution of the rotational forces must be considered as well as distribution of the transnational forces. If rigid or semirigid diaphragms are used, the designer may assume that torsional forces are distributed to the shear walls in proportion to their relative rigidities and their distances from the center of rigidity. A flexible diaphragm should not be considered capable of distributing torsional forces.

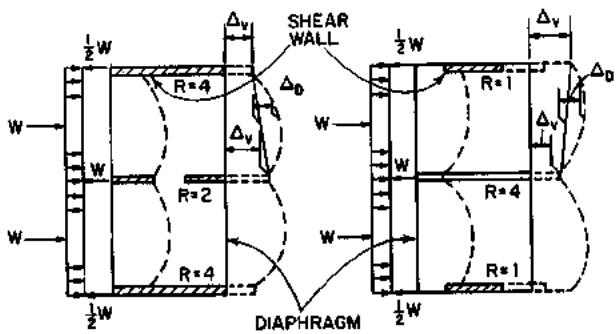


FIGURE 5.84 Horizontal section through shear walls connected by a flexible diaphragm.

See also Art. 5.12.5.

Example of Torque Distribution to Shear Walls. To illustrate load-distribution calculations for shear walls with rigid or semirigid diaphragms, Fig. 5.85 shows a horizontal section through three shear walls *A*, *B*, and *C* taken above a rigid floor. Wall *B* is 16 ft from wall *A*, and 24 ft from wall *C*. Rigidity of *A* 0.33, of *B* 0.22, and of *C* 0.45 (Art. 5.12.5). A 20-kip horizontal force acts at floor level parallel to the shear walls and midway between *A* and *C*.

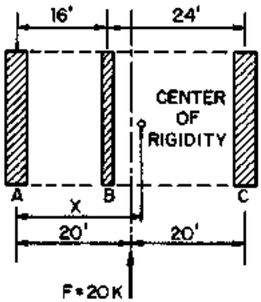


FIGURE 5.85 Rigid diaphragm distributes 20-kip horizontal force to shear walls *A*, *B*, and *C*.

The center of rigidity of the shear walls is located, relative to wall *A*, by taking moments about *A* of the wall rigidities and dividing the sum of these moments by the sum of the wall rigidities, in this case 1.00.

$$x = 0.22 \times 16 + 0.45 \times 40$$

$$= 21.52 \text{ ft}$$

Thus, the 20-kip lateral force has an eccentricity of $21.52 - 20 = 1.52$ ft. The eccentric force may be resolved into a 20-kip force acting through the center of rigidity and not producing torque, and a couple producing a torque of $20 \times 1.52 = 30.4$ ft-kips.

The nonrotational force is distributed to the shear walls in proportion to their rigidities:

Wall *A*: $0.33 \times 20 = 6.6$ kips

Wall *B*: $0.22 \times 20 = 4.4$ kips

Wall *C*: $0.45 \times 20 = 9.0$ kips

For distribution of the torque to the shear walls, the equivalent of moment of inertia must first be computed:

$$I = 0.33(21.52)^2 + 0.22(5.52)^2 + 0.45(18.48)^2 = 313$$

Then, the torque is distributed in direct proportion to shear-wall rigidity and distance from center of rigidity and in inverse proportion to *I*.

Wall *A*: $30.4 \times 0.33 \times 21.52/313 = 0.690$ kips

Wall *B*: $30.4 \times 0.22 \times 5.52/313 = 0.118$ kips

Wall *C*: $30.4 \times 0.45 \times 18.48/313 = 0.808$ kips

The torsional forces should be added to the nonrotational forces acting on walls *A* and *B*, whereas the torsional force on wall *C* acts in the opposite direction to the nonrotational force. For a conservative design, the torsional force on wall *C* should not be subtracted. Hence, the walls should be designed for the following forces:

$$\text{Wall } A\text{: } 6.6 + 0.7 = 7.3 \text{ kips}$$

$$\text{Wall } B\text{: } 4.4 + 0.1 = 4.5 \text{ kips}$$

$$\text{Wall } C\text{: kips}$$

5.12.3 Deflections of Bents or Shear Walls

When parallel bents or shear walls are connected by rigid diaphragms (Art. 5.12.1) and horizontal loads are distributed to the vertical resisting elements in proportion to their relative rigidities, the relative rigidity of the framing depends on the combined horizontal deflections due to shear and flexure. For the dimensions of lateral-force-resisting framing used in many high-rise buildings, however, deflections due to flexure greatly exceed those due to shear. In such cases, only flexural rigidity need be considered in determination of relative rigidity of the bents and shear walls (Art. 5.12.5).

Horizontal deflections can be determined by treating the bents and shear walls as cantilevers. Deflections of braced bents can be calculated by the dummy-unit-load method (Art. 5.10.4) or a matrix method (Art. 5.13.3). Deflections of rigid frames can be obtained by summing the drifts of the stories, as determined by moment distribution (Art. 5.11.9) or a matrix method. And deflections of shear walls can be computed from formulas given in Art. 5.5.15, the dummy-unit-load method, or a matrix method.

For a shear wall with a solid, rectangular cross section, the flexural deflection at the top under uniform loading is given by the formula for a cantilever in Fig. 5.39:

$$\delta_c = \frac{wH^4}{8EI} \tag{5.142}$$

where w = uniform lateral load
H = height of the wall
E = modulus of elasticity of the wall material
I = moment of inertia of wall cross section = $tL^3/12$
t = wall thickness
L = length of wall

The cantilever shear deflection under uniform loading may be computed from

$$\delta_v = \frac{0.6wH^2}{E_v A} \tag{5.143}$$

where E_v = modulus of rigidity of wall cross section
= $E/2(1 + \mu)$
μ = Poisson's ratio for the wall material (0.25 for concrete and masonry)
A = cross-sectional area of the wall = tL

The total deflection then is

$$\delta_c + \delta_v = \frac{1.5wH}{Et}\left[\left(\frac{H}{L}\right)^3 + \frac{H}{L}\right] \tag{5.144}$$

For a cantilever wall subjected to a concentrated load P at the top, the flexural deflection at the top is

$$\delta_c = \frac{PH^3}{3EI} \tag{5.145}$$

The shear deflection at the top of the wall is

$$\delta_v = \frac{1.2PH}{E_v A} \tag{5.146}$$

Hence, the total deflection of the cantilever is

$$\delta = \frac{4P}{Et}\left[\left(\frac{H}{L}\right)^3 + 0.75\,\frac{H}{L}\right] \tag{5.147}$$

For a wall fixed against rotation a the top and subjected to a concentrated load P at the top, the flexural deflection at the top is

$$\delta_c = \frac{PH^3}{12EI} \tag{5.148}$$

The shear deflection for the fixed-end wall is given by Eq. (5.145). Hence, the total deflection for the wall is

$$\delta = \frac{P}{Et}\left[\left(\frac{H}{L}\right)^3 + 3\,\frac{H}{L}\right] \tag{5.149}$$

5.12.4 Diaphragm-Deflection Limitations

As indicated in Art. 5.12.1, horizontal deflection of diaphragms plays an important role in determining lateral-load distribution to bents and shear walls. Another design consideration is the necessity of limiting diaphragm deflection to prevent excessive stresses in walls perpendicular to shear walls. Equation (5.150) was suggested by the Structural Engineers Association of Southern California for allowable story deflection Δ, in, of masonry or concrete building walls.

$$\Delta = \frac{h^2 f}{0.01Et} \tag{5.150}$$

where h = height of wall between adjacent horizontal supports, ft
t = thickness of wall, in
f = allowable flexural compressive stress of wall material, psi
E = modulus of elasticity of wall material, psi

This limit on deflection must be applied with engineering judgment. For example, continuity of wall at floor level is assumed, and in many cases is not present because of through-wall flashing. In this situation, the deflection may be based on the allowable compressive stress in the masonry, if a reduced cross section of wall is assumed. The effect of reinforcement, which may be present in a reinforced brick masonry wall or as a tie to the floor system in a nonreinforced or partly reinforced masonry wall, was not considered in development of Eq. (5.150). Note also that the limit on wall deflection is actually a limit on differential deflection between two successive floor, or diaphragm, levels.

Maximum span-width or span-depth ratios for diaphragms are usually used to control horizontal diaphragm deflection indirectly. Normally, if the diaphragm is designed with the proper ratio, the diaphragm deflection will not be critical. Table 5.8 may be used as a guide for proportioning diaphragms.

5.12.5 Shear-Wall Rigidity

Where shear walls are connected by rigid diaphragms so that they must deflect equally under horizontal loads, the proportion of total horizontal load at any level carried by a shear wall parallel to the load depends on the relative rigidity, or stiffness, of the wall in the direction of the load (Art. 5.12.1). Rigidity of a shear wall is inversely proportional to its deflection under unit horizontal load. This deflection equals the sum of the shear and flexural deflections under the load (Art. 5.12.3).

Where a shear wall contains no openings, computations for deflection and rigidity are simple. In Fig. 5.86*a*, each of the shear walls has the same length and rigidity. So each takes half the total load. In Fig. 5.86*b*, length of wall *C* is half that of wall *D*. By Eq. (5.142), *C* therefore receives less than one-eighth the total load.

Walls with Openings. Where shear walls contain openings, such as doors and windows, computations for deflection and rigidity are more complex. But approximate methods may be used.

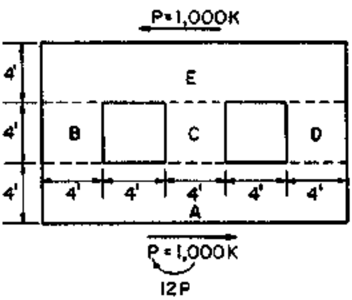


FIGURE 5.87 Shear wall, 8 in thick, with openings.

For example, the wall in Fig. 5.87, subjected to a 1000-kip load at the top, may be treated in parts. The wall is 8 in thick, and its modulus of elasticity $E = 2400$ ksi. Its height-length ratio H/L is $^{12}\!/_{20} = 0.6$. The wall is perforated by two, symmetrically located, 4-ft-square openings.

Deflection of this wall can be estimated by subtracting from the deflection it would have if it were solid the deflection of a solid, 4-ft-deep, horizontal midstrip, and then adding the deflection of the three coupled piers *B*, *C*, and *D*.

Deflection of the 12-ft-high solid wall can be obtained from Eq. (5.147):

$$\delta = \frac{4 \times 10^3}{2.4 \times 10^3 \times 8} [(0.6)^3 + 0.75 \times 0.6] = 0.138 \text{ in}$$

Rigidity of the solid wall then is

$$R = \frac{1}{0.138} = 7.22$$

Similarly, the deflection of the 4-ft-deep solid midstrip can be computed from Eq. (5.147), with $H/L = {}^{4}\!/_{20} = 0.20$.

$$\delta = \frac{4 \times 10^3}{2.4 \times 10^3 \times 8} [(0.20)^3 + 0.75 \times 0.20] = 0.033 \text{ in}$$

Deflection of the piers, which may be considered fixed top and bottom, can be

TABLE 5.8 Maximum Span-Width or Span-Depth Ratios for diaphragms—Roofs or Floors*

Diaphragm construction	Masonry and concrete walls	Wood and light steel walls
Concrete	Limited by deflection	
Steel deck (continuous sheet in a single plane)	4:1	5:1
Steel deck (without continuous sheet)	2:1	2½:1
Cast-in-place reinforced gypsum roofs	3:1	4:1
Plywood (nailed all edges)	3:1	4:1
Plywood (nailed to supports only—blocking may be omitted between joists)	2½:1	3½:1
Diagonal sheating (special)	3:1†	3½:1
Diagonal sheating (conventional construction)	2:1†	2½:1

* From California Administrative code, Title 21, Public Works.

† Use of diagonal sheathed or unblocked plywood diaphragms for buildings having masonry or reinforced concrete walls shall be limited to one-story buildings or to the roof of a top story.

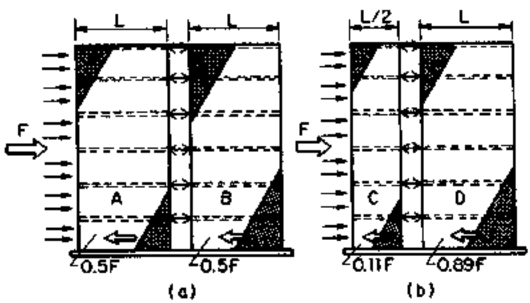


FIGURE 5.86 Distribution of horizontal load to parallel shear walls: (*a*) walls with the same length and rigidity share the load equally; (*b*) wall half the length of another carries less than one-eighth of the load.

obtained from Eq. (5.149), with $H/L = 4/4 = 1$. For any one of the piers, the deflection is

$$\delta' v = \frac{10^3}{2.4 \times 10^3 \times 8} (1 + 3) = 0.208 \text{ in}$$

The rigidity of a single pier is $1/0.208 = 4.81$, and of the three piers, $3 \times 4.81 = 14.43$. Therefore, the deflection of the three piers when coupled is

$$\delta = \frac{1}{14.43} = 0.069 \text{ in}$$

The deflection of the whole wall, with openings, then is approximately

$$\delta = 0.138 - 0.033 + 0.069 = 0.174 \text{ in}$$

And its rigidity is

$$R = \frac{1}{0.174} = 5.74$$

5.12.6 Effects of Shear-Wall Arrangements

To increase the stiffness of shear walls and thus their resistance to bending, intersecting walls or flanges may be used. Often in the design of buildings, A-, T-, U-, L-, and I-shaped walls in plan develop as natural parts of the design. Shear walls with these shapes have better flexural resistance than a single, straight wall.

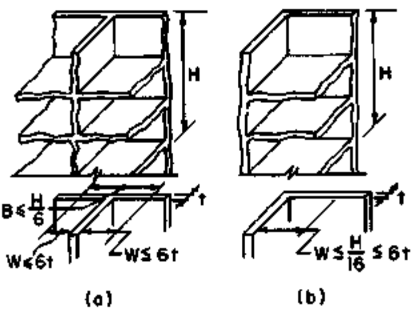


FIGURE 5.88 Effective flange width of shear walls may be less than the actual width: (*a*) limits for flanges of I and T shapes; (*b*) limits for C and L shapes.

In calculation of flexural stresses in masonry shear walls for symmetrical T or I sections, the effective flange width may not exceed one-sixth the total wall height above the level being analyzed. For unsymmetrical L or C sections, the width considered effective may not exceed one-sixteenth the total wall height above the level being analyzed. In either case, the overhang for any section may not exceed six times the flange thickness (Fig. 5.88).

The shear stress at the intersection of the walls should not exceed the permissible shear stress.

5.12.7 Coupled Shear Walls

Another method than that described in Art. 5.12.6 for increasing the stiffness of a bearing-wall structure and reducing the possibility of tension developing in masonry shear walls under lateral loads is coupling of coplanar shear walls.

Figure 5.89 and 5.90 indicate the effect of coupling on stress distribution in a pair of walls under horizontal forces parallel to the walls. A flexible connection between the walls is assumed in Figs. 5.89*a* and 5.90*a*, so that the walls act as independent vertical cantilevers in resisting lateral loads. In Figs. 5.89*b* and 5.90*b*, the walls are assumed to be connected with a more rigid member, which is capable of shear and moment transfer. A rigid-frame type action results. This can be accomplished with a steel-reinforced concrete, or reinforced brick masonry coupling.

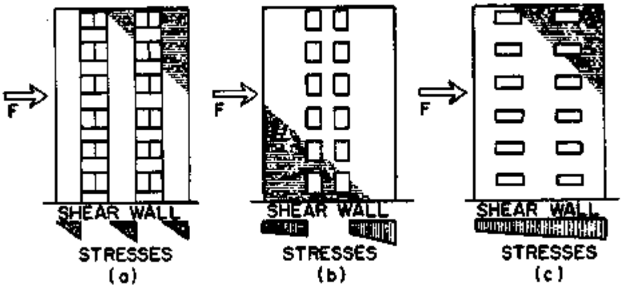


FIGURE 5.89 Stress distribution in end shear walls: (*a*) with flexible coupling; (*b*) with rigid-frame-type action; (*c*) with plate-type action.

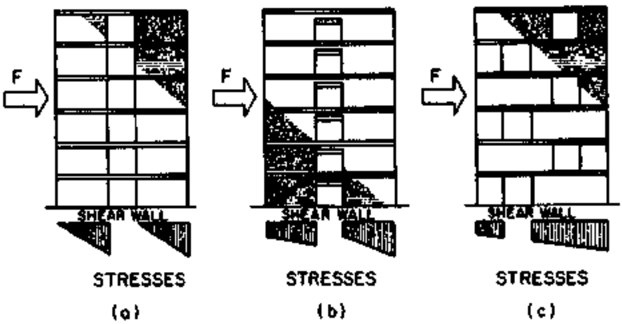


FIGURE 5.90 Stress distribution in interior shear walls: (*a*) with flexible coupling; (*b*) with rigid-frame-type action; (*c*) with plate-type action.

A plate-type action is indicated in Figs. 5.89*c* and 5.90*c*. This assumes an extremely rigid connection between walls, such as fully story-height walls or deep rigid spandrels.

5.13 FINITE-ELEMENT METHODS

From the basic principles given in preceding articles, systematic procedures have been developed for determining the behavior of a structure from a knowledge of the behavior under load of its components. In these methods, called finite-element methods, a structural system is considered an assembly of a finite number of finite-size components, or elements. These are assumed to be connected to each other only at discrete points, called nodes. From the characteristics of the elements, such as their stiffness or flexibility, the characteristics of the whole system can be derived. With these known, internal stresses and strains throughout can be computed.

Choice of elements to be used depends on the type of structure. For example, for a truss with joints considered hinged, a natural choice of element would be a bar, subjected only to axial forces. For a rigid frame, the elements might be beams subjected to bending and axial forces, or to bending, axial forces, and torsion. For a thin plate or shell, elements might be triangles or rectangles, connected at vertices. For three-dimensional structures, elements might be beams, bars, tetrahedrons, cubes, or rings.

For many structures, because of the number of finite elements and nodes, analysis by a finite-element method requires mathematical treatment of large amounts of data and solution of numerous simultaneous equations. For this purpose, the use of computers is advisable. The mathematics of such analyses is usually simpler and more compact when the data are handled in matrix for. (See also Art. 5.10.7.)

5.13.1 Force and Displacement Methods

The methods used for analyzing structures generally may be classified as force (flexibility) or displacement (stiffness) methods.

In analysis of statically indeterminate structures by force methods, forces are chosen as redundants, or unknowns. The choice is made in such a way that equilibrium is satisfied. These forces are then determined from the solution of equations that ensure compatibility of all displacements of elements at each node. After the redundants have been computed, stresses and strains throughout the structure can be found from equilibrium equations and stress-strain relations.

In displacement methods, displacements are chosen as unknowns. The choice is made in such a way that geometric compatibility is satisfied. These displacements are then determined from the solution of equations that ensure that forces acting at each node are in equilibrium. After the unknowns have been computed, stresses and stains throughout the structure can be found from equilibrium equations and stress-strain relations.

In choosing a method, the following should be kept in mind: In force methods, the number of unknowns equals the degree of indeterminacy. In displacement methods, the number of unknowns equals the degrees of freedom of displacement at nodes. The fewer the unknowns, the fewer the calculations required.

Both methods are based on the force-displacement relations and utilize the stiffness and flexibility matrices described in Art. 5.10.7. In these methods, displacements and external forces are resolved into components—usually horizontal, vertical, and rotational—at nodes, or points of connection of the finite elements. In accordance with Eq. (5.103*a*), the stiffness matrix transforms displacements into forces. Similarly, in accordance with Eq. (5.103*b*), the flexibility matrix transforms forces into displacements. To accomplish the transformation, the nodal forces and displacements must be assembled into correspondingly positioned elements of force and displacement vectors. Depending on whether the displacement or the force method is chosen, stiffness or flexibility matrices are then established for each of the finite elements and these matrices are assembled to form a square matrix, from which the stiffness or flexibility matrix for the structure as a whole is derived. With that matrix known and substituted into equilibrium and compatibility equations for the structure, all nodal forces and displacements of the finite elements can be determined from the solution of the equations. Internal stresses and strains in the elements can be computed from the now known nodal forces and displacements.

5.13.2 Element Flexibility and Stiffness Matrices

The relationship between *independent* forces and displacements at nodes of finite elements comprising a structure is determined by flexibility matrices **f** or stiffness matrices **k** of the elements. In some cases, the components of these matrices can be developed from the defining equations:

The jth column of a flexibility matrix of a finite element contains all the nodal displacements of the element when one force $\mathbf{S}_j$ is set equal to unity and all other independent forces are set equal to zero.

The jth column of a stiffness matrix of a finite element consists of the forces acting at the nodes of the element to produce a unit displacement of the node at which displacement δ_j occurs and in the direction of δ_j but no other nodal displacements of the element.

Bars with Axial Stress Only. As an example of the use of the definitions of flexibility and stiffness, consider the simple case of an elastic bar under tension applied by axial forces P_i and P_j at nodes i and j, respectively (Fig. 5.91). The bar might be the finite element of a truss, such as a diagonal or a hanger. Connections to other members are made at nodes i and j, which an transmit only forces in the directions i to j or j to i.

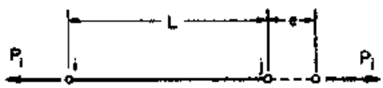


FIGURE 5.91 Elastic bar in tension.

For equilibrium, $P_i = P_j = P$. Displacement of node j relative to node i is e. From Eq. (5.23), $e = PL/AE$, where L is the initial length of the bar, A the bar cross-sectional area, and E the modulus of elasticity. Setting P eq 1 yields the flexibility of the bar,

$$f = \frac{L}{AE} \tag{5.151}$$

Setting $e = 1$ gives the stiffness of the bar,

$$k = \frac{AE}{L} \tag{5.152}$$

Beams with Bending Only. As another example of the use of the definition to determine element flexibility and stiffness matrices, consider the simple case of an elastic prismatic beam in bending applied by moments M_i and M_j at nodes i and j, respectively (Fig. 5.92a). The beam might be a finite element of a rigid frame. Connections to other members are made at nodes i and j, which can transmit moments and forces normal to the beam.

Nodal displacements of the element can be sufficiently described by rotations θ_i and θ_j relative to the straight line between nodes i and j. For equilibrium, forces $V_j = -V_i$ normal to the beam are required at nodes j and i, respectively, and $V_j = (M_i + M_j)/L$, where L is the span of the beam. Thus, M_i and M_j are the only

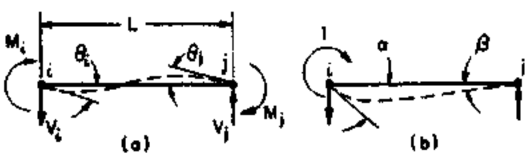


FIGURE 5.92 Beam subjected to end moments and shears.

independent forces acting. Hence, the force-displacement relationship can be written for this element as

$$\theta = \begin{bmatrix} \theta_i \\ \theta_j \end{bmatrix} = \mathbf{f} \begin{bmatrix} M_i \\ M_j \end{bmatrix} = \mathbf{fM} \tag{5.153}$$

$$\mathbf{M} = \begin{bmatrix} M_i \\ M_j \end{bmatrix} = \mathbf{k} \begin{bmatrix} \theta_i \\ \theta_j \end{bmatrix} = \mathbf{k}\theta \tag{5.154}$$

The flexibility matrix **f** then will be a 2×2 matrix. The first column can be obtained by setting $M_i = 1$ and $M_j = 0$ (Fig. 5.92*b*). The resulting angular rotations are given by Eqs. (5.107) and (5.108): For a beam with constant moment of inertia I and modulus of elasticity E, the rotations are $\alpha = L/3EI$ and $\beta = -L/6EI$. Similarly, the second column can be developed by setting $M_i = 0$ and $M_j = 1$.

The flexibility matrix for a beam in bending then is

$$\mathbf{f} = \begin{bmatrix} \dfrac{L}{3EI} & -\dfrac{L}{6EI} \\ -\dfrac{L}{6EI} & \dfrac{L}{3EI} \end{bmatrix} = \frac{L}{6EI} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \tag{5.155}$$

The stiffness matrix, obtained in a similar manner or by inversion of **f**, is

$$\mathbf{k} = \begin{bmatrix} \dfrac{4EI}{L} & \dfrac{2EI}{L} \\ \dfrac{2EI}{L} & \dfrac{4EI}{L} \end{bmatrix} = \frac{2EI}{L} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{5.156}$$

Beams Subjected to Bending and Axial Forces. For a beam subjected to nodal moments M_i and M_j and axial forces P, flexibility and stiffness are represented by 3×3 matrices. The load-displacement relations for a beam of span L, constant moment of inertia I, modulus of elasticity E, and cross-sectional area A are given by

$$\begin{bmatrix} \theta \\ \theta_j \\ e \end{bmatrix} = \mathbf{f} \begin{bmatrix} M_i \\ M_j \\ P \end{bmatrix} \qquad \begin{bmatrix} M_i \\ M_j \\ P \end{bmatrix} = \mathbf{k} \begin{bmatrix} \theta_i \\ \theta_j \\ e \end{bmatrix} \tag{5.157}$$

In this case, the flexibility matrix is

$$\mathbf{f} = \frac{L}{6EI} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & \eta \end{bmatrix} \tag{5.158}$$

where $\eta = 6I/A$, and the stiffness matrix is

$$\mathbf{k} = \frac{EI}{L} \begin{bmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & \psi \end{bmatrix} \tag{5.159}$$

where $\psi = A/I$.

5.13.3 Displacement (Stiffness) Method

With the stiffness or flexibility matrix of each finite element of a structure known, the stiffness or flexibility matrix for the whole structure can be determined, and with that matrix, forces and displacements throughout the structure can be computed (Art. 5.13.2). To illustrate the procedure, the steps in the displacement, or stiffness, method are described in the following. The steps in the flexibility method are similar. For the stiffness method:

Step 1. Divide the structure into interconnected elements and assign a number, for identification purposes, to every node (intersection and terminal of elements). It may also be useful to assign an identifying number to each element.

Step 2. Assume a right-handed cartesian coordinate system, with axes x, y, z. Assume also at each node of a structure to be analyzed a system of base unit vectors, $\mathbf{e}_1$ in the direction of the x axis, $\mathbf{e}_2$ in the direction of the y axis, and $\mathbf{e}_3$ in the direction of the z axis. Forces and moments acting at a node are resolved into components in the directions of the base vectors. Then, the forces and moments at the node may be represented by the vector $P_i\mathbf{e}_i$, where P_i is the magnitude of the force or moment acting in the direction of $\mathbf{e}_i$. This vector, in turn, may be conveniently represented by a column matrix $\mathbf{P}$. Similarly, the displacements—translations and rotation—of the node may be represented by the vector $\Delta_i\mathbf{e}_i$, where Δ_i is the magnitude of the displacement acting in the direction of $\mathbf{e}_i$. This vector, in turn, may be represented by a column matrix Δ.

For compactness, and because, in structural analysis, similar operations are performed on all nodal forces, all the loads, including moments, acting on all the nodes may be combined into a single column matrix $\mathbf{P}$. Similarly, all the nodal displacements may be represented by a single column matrix Δ.

When loads act along a beam, they should be replaced by equivalent forces at the nodes—simple-beam reactions and fixed-end moments, both with signs reversed from those induced by the loads. The final element forces are then determined by adding these moments and reactions to those obtained from the solution with only the nodal forces.

Step 3. Develop a stiffness matrix $\mathbf{k}_i$ for each element i of the structure (see Art. 5.13.2). By definition of stiffness matrix, nodal displacements and forces for the i the element are related by

$$\mathbf{S}_i = \mathbf{k}_i\boldsymbol{\delta}_i \quad i = 1, 2, \ldots, n \tag{5.160}$$

where $\mathbf{S}_i$ = matrix of forces, including moments and torques acting at the nodes of the ith element

$\boldsymbol{\delta}_i$ = matrix of displacements of the nodes of the ith element

Step 4. For compactness, combine this relationship between nodal displacements and forces for each element into a single matrix equation applicable to all the elements:

$$\mathbf{S} = \mathbf{k}\boldsymbol{\delta} \tag{5.161}$$

where $\mathbf{S}$ = matrix of all forces acting at the nodes of all elements

$\boldsymbol{\delta}$ = matrix of all nodal displacements for all elements

$$\mathbf{k} = \begin{bmatrix} \mathbf{k}_1 & 0 & \dots & 0 \\ 0 & \mathbf{k}_2 & \dots & 0 \\ .. & .. & \dots & .. \\ 0 & 0 & \dots & \mathbf{k}_n \end{bmatrix} \tag{5.162}$$

Step 5. Develop a matrix b_0 that will transform the displacements Δ of the nodes of the structure into the displacement vector $\boldsymbol{\delta}$ while maintaining geometric compatibility:

$$\boldsymbol{\delta} = \mathbf{b}_0 \boldsymbol{\Delta} \tag{5.163}$$

$\mathbf{b}_0$ is a matrix of influence coefficients. The jth column of $\mathbf{b}_0$ contains the element nodal displacements when the node where Δ_j occurs is given a unit displacement in the direction of Δ_j, and no other nodes are displaced.

Step 6. Compute the stiffness matrix **K** for the whole structure from

$$\mathbf{K} = \mathbf{b}_0^T \mathbf{k} \mathbf{b}_0 \tag{5.164}$$

where $\mathbf{b}_0^T$ = transpose of $\mathbf{b}_0$ = matrix $\mathbf{b}_0$ with rows and columns interchanged

This equation may be derived as follows: From energy relationship, $\mathbf{P} = \mathbf{b}_0^T\mathbf{S}$. Substitution of $\mathbf{k}\boldsymbol{\delta}$ for $\mathbf{S}$ [Eq. (5.161)] and then substitution of $\mathbf{b}_0\boldsymbol{\Delta}$ for $\boldsymbol{\delta}$ [Eq. (5.163)] yields $\mathbf{P} = \mathbf{b}_0^T\mathbf{k}\mathbf{b}_0\boldsymbol{\Delta}$. Comparison of this with Eq. (5.103*a*), $\mathbf{P} = \mathbf{k}\boldsymbol{\Delta}$ leads to Eq. (5.164).

Step 7. With the stiffness matrix **K** now known, solve the simultaneous equations

$$\boldsymbol{\Delta} = \mathbf{K}^{-1}\mathbf{P} \tag{5.165}$$

for the nodal displacements $\boldsymbol{\Delta}$. With these determined, calculate the member forces from

$$\mathbf{S} = \mathbf{k}\mathbf{b}_0\boldsymbol{\Delta} \tag{5.166}$$

(N. M. Baran, "Finite Element Analysis on Microcomputers," and H. Kardestuncer and D. H. Norris, "Finite Element Handbook," McGraw-Hill Publishing Company, New York; K. Bathe, "Finite Element Procedures in Engineering Analysis," T. R. Hughes, "The Finite Element Method," W. Weaver, Jr., and P. R. Johnston, "Structural Dynamics by Finite Elements," and H. T. Y. Yang, "Finite Element Structural Analysis," Prentice-Hall, Englewood Cliffs, N.J.)

5.14 STRESSES IN ARCHES

An arch is a curved beam, the radius of curvature of which is very large relative to the depth of the section. It differs from a straight beam in that: (1) loads induce both bending and direct compressive stresses in an arch; (2) arch reactions have horizontal components even though loads are all vertical; and (3) deflections have horizontal as well as vertical components (see also Arts. 5.6.1 to 5.6.4). Names of arch parts are given in Fig. 5.93.

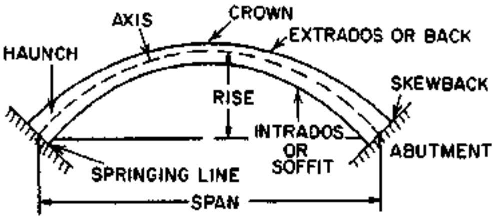


FIGURE 5.93 Components of an arch.

The necessity of resisting the horizontal components of the reactions is an important consideration in arch design. Sometimes these forces are taken by the tie rods between the supports, sometimes by heavy abutments or buttresses.

Arches may be built with fixed ends, as can straight beams, or with hinges at the supports. They may also be built with a hinge at the crown.

5.14.1 Three-Hinged Arches

An arch with a hinge at the crown as well as at both supports (Fig. 5.94) is statically determinate. There are four unknowns—two horizontal and two vertical components of the reactions—but four equations based on the laws of equilibrium are available: (1) The sum of the horizontal forces must be zero. (2) The sum of the moments about the left support must be zero. (3) The sum of the moments about the right support must be zero. (4) The bending moment at the crown hinge must be zero (not to be confused with the sum of the moments about the crown, which also must be equal to zero but which would not lead to an independent equation for the solution of the reactions).

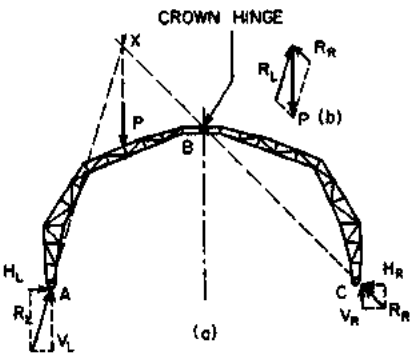


FIGURE 5.94 Three-hinged arch.

Stresses and reactions in three-hinged arches can be determined graphically by taking advantage of the fact that the bending moment at the crown hinge is zero. For example, in Fig. 5.94*a*, a concentrated load *P* is applied to segment *AB* of the arch. Then, since the bending moment at *B* must be zero, the line of action of the reaction at *C* must pass through the crown hinge. It intersects the line of action of *P* at *X*. The line of action of the reaction at *A* must also pass through *X*. Since *P* is equal to the sum of the reactions, and since the directions of the reactions have thus been determined, the magnitude of the reactions can be measured from a parallelogram of forces (Fig. 5.94*b*). When the reactions have been found, the stresses can be computed from the laws of statics (see Art. 5.14.3) or, in the case of a trussed arch, determined graphically.

5.14.2 Two-Hinged Arches

When an arch has hinges at the supports only (Fig. 5.95), it is statically indeterminate, and some knowledge of its deformations is required to determine the reactions. One procedure is to assume that one of the supports is on rollers. This makes the arch statically determinate. The reactions and the horizontal movement of the support are computed for this condition (Fig. 5.95*b*). Then, the magnitude of the horizontal force required to return the movable support to its original position is calculated (Fig. 5.95*c*). The reactions for the two-hinged arch are finally found by superimposing the first set of reactions on the second (Fig. 5.95*d*).

For example, if δx is the horizontal movement of the support due to the loads, and if $\delta x'$ is the horizontal movement of the support due to a unit horizontal force applied to the support, then

$$\delta x + H\delta x' = 0 \tag{5.167}$$

$$H = -\frac{\delta x}{\delta x'} \tag{5.168}$$

where H is the unknown horizontal reaction. (When a tie rod is used to take the thrust, the right-hand side of Eq. (5.167) is not zero, but the elongation of the rod, HL/AE.) The dummy unit-load method [Eq. (5.96)] can be used to compute δx and $\delta x'$:

$$\delta x = \int_A^B \frac{My}{EI}\,ds - \int_A^B \frac{N\,dx}{AE} \tag{5.169}$$

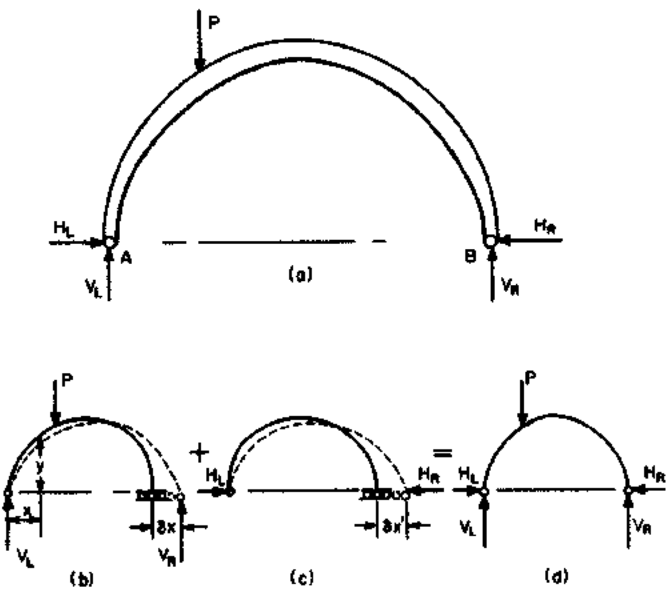


FIGURE 5.95 Two-hinged arch.

where M = moment at any section resulting from loads
N = normal thrust on cross section
A = cross-sectional area of arch
y = ordinate of section measured from A as origin, when B is on rollers
I = moment of inertia of section
E = modulus of elasticity
ds = differential length along axis of arch
dx = differential length along horizontal

$$\delta x' = -\int_A^B \frac{y^2}{EI}\,ds - \int_A^B \frac{\cos^2\alpha\,dx}{AE} \tag{5.170}$$

where α = the angle the tangent to the axis at the section makes with the horizontal. Unless the thrust is very large and would be responsible for large strains in the direction of the arch axis, the second term on the right-hand side of Eq. (5.169) can usually be ignored.

In most cases, integration is impracticable. The integrals generally must be evaluated by approximate methods. The arch axis is divided into a convenient number of sections and the functions under the integral sign evaluated for each section. The sum is approximately equal to the integral. Thus, for the usual two-hinged arch,

$$H = \frac{\sum_A^B (My\,\Delta s/EI)}{\sum_A^B (y^2\,\Delta s/EI) + \sum_A^B (\cos^2\alpha\,\Delta x/AE)} \tag{5.171}$$

(S. Timoshenko and D. H. Young, "Theory of Structures," McGraw-Hill Book Company, New York; S. F. Borg and J. J. Gennaro, "Modern Structural Analysis," Van Nostrand Reinhold Company, Inc., New York.)

5.14.3 Stresses in Arch Ribs

When the reactions have been found for an arch (Arts. 5.14.1 to 5.14.2), the principal forces acting on any cross section can be found by applying the equations of equilibrium. For example, consider the portion of an arch in Fig. 5.96, where the

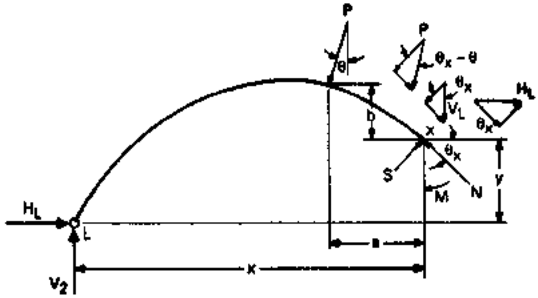


FIGURE 5.96 Interior stresses at X hold portion LX of an arch rib in equilibrium.

forces acting at an interior section X are to be found. The load P, H_L (or H_R), and V_L (or V_R) may be resolved into components parallel to the axial thrust N and the shear S at X, as indicated in Fig. 5.96. Then, by equating the sum of the forces in each direction to zero, we get

$$N = V_L \sin \theta_x + H_L \cos \theta_x + P \sin (\theta_x - \theta) \tag{5.172}$$

$$S = V_L \cos \theta_x - H_L \sin \theta_x + P \cos (\theta_x - \theta) \tag{5.173}$$

And the bending moment at X is

$$M = V_L x - H_1 y - Pa \cos \theta - Pb \sin \theta \tag{5.174}$$

The shearing unit stress on the arch cross section at X can be determined from S wit the aid of Eq. (5.59). The normal unit stresses can be calculated from N and M with the aid of Eq. (5.67).

In designing an arch, it may be necessary to compute certain secondary stresses, in addition to those caused by live, dead, wind, and snow loads. Among the secondary stresses to be considered are those due to temperature changes, rib shortening due to thrust or shrinkage, deformation of tie rods, and unequal settlement of footings. The procedure is the same as for loads on the arch, with the deformations producing the secondary stresses substituted for or treated the same as the deformations due to loads.

5.15 THIN-SHELL STRUCTURES

A structural membrane or shell is a curved surface structure. Usually, it is capable of transmitting loads in more than two directions to supports. It is highly efficient structurally when it is so shaped, proportioned, and supported that it transmits the loads without bending or twisting.

A membrane or a shell is defined by its middle surface, halfway between its extrados, or outer surface and intrados, or inner surface. Thus, depending on the geometry of the middle surface, it might be a type of dome, barrel arch, cone, or hyperbolic paraboloid. Its thickness is the distance, normal to the middle surface, between extrados and intrados.

5.15.1 Thin-Shell Analysis

A thin shell is a shell with a thickness relatively small compared with its other dimensions. But it should not be so thin that deformations would be large compared with the thickness.

The shell should also satisfy the following conditions: Shearing stresses normal to the middle surface are negligible. Points on a normal to the middle surface before it is deformed lie on a straight line after deformation. And this line is normal to the deformed middle surface.

Calculation of the stresses in a thin shell generally is carried out in two major steps, both usually involving the solution of differential equations. In the first, bending and torsion are neglected (membrane theory, Art. 5.15.2). In the second step, corrections are made to the previous solution by superimposing the bending and shear stresses that are necessary to satisfy boundary conditions (bending theory, Art. 5.15.3).

Ribbed Shells. For long-span construction, thin shells often are stiffened at intervals by ribs. Usually, the construction is such that the shells transmit some of the load imposed on them to the ribs, which then perform structurally as more than just stiffeners. Stress and strain distributions in shells and ribs consequently are complicated by the interaction between shells and ribs. The shells restrain the ribs, and the ribs restrain the shells. Hence, ribbed shells usually are analyzed by approximate methods based on reasonable assumptions.

For example, for a cylindrical shell with circumferential ribs, the ribs act like arches. For an approximate analysis, the ribbed shell therefore may be assumed to be composed of a set of arched ribs with the thin shell between the ribs acting in the circumferential direction as flanges of the arches. In the longitudinal direction, it may be assumed that the shell transfers load to the ribs in flexure. Designers may adjust the results of a computation based on such assumptions to correct for a variety of conditions, such as the effects of free edges of the shell, long distances between ribs, relative flexibility of ribs and shell, and characteristics of the structural materials.

5.15.2 Membrane Theory for Thin Shells

Thin shells usually are designed so that normal shears, bending moments, and torsion are very small, except in relatively small portions of the shells. In the membrane theory, these stresses are ignored.

Despite the neglected stresses, the remaining stresses ae in equilibrium, except possibly at boundaries, supports, and discontinuities. At any interior point, the number of equilibrium conditions equals the number of unknowns. Thus, in the membrane theory, a thin shell is statically determinate.

The membrane theory does not hold for concentrated loads normal to the middle surface, except possibly at a peak or valley. The theory does not apply where boundary conditions are incompatible with equilibrium. And it is in exact where there is geometric incompatibility at the boundaries. The last is a common condition, but the error is very small if the shell is not very flat. Usually, disturbances of membrane equilibrium due to incompatibility with deformations at boundaries, supports, or discontinuities are appreciable only in a narrow region about each source of disturbance. Much larger disturbances result from incompatibility with equilibrium conditions.

To secure the high structural efficiency of a thin shell, select a shape, proportions, and supports for the specific design conditions that come as close as possible to satisfying the membrane theory. Keep the thickness constant; if it must change, use a gradual taper. Avoid concentrated and abruptly changing loads. Change curvature gradually. Keep discontinuities to a minimum. Provide reactions that are tangent to the middle surface. At boundaries, ensure, to the extent possible, compatibility of shell deformations with deformations of adjoining members, or at least keep restraints to a minimum. Make certain that reactions along boundaries are equal in magnitude and direction to the shell forces there.

Means usually adopted to satisfy these requirements at boundaries and supports are illustrated in Fig. 5.97. In Fig. 5.97*a*, the slope of the support and provision for movement normal to the middle surface ensure a reaction tangent to the middle surface. In Fig. 5.97*b*, a stiff rib, or ring girder, resists unbalanced shears and

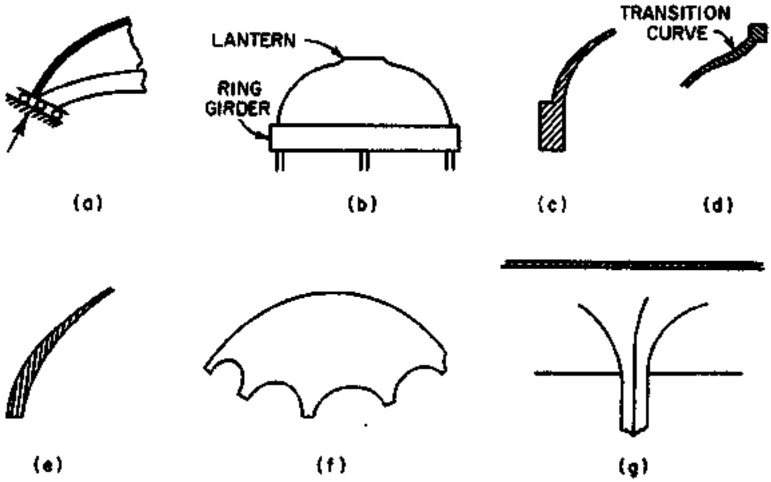


FIGURE 5.97 Special provisions made at supports and boundaries of thin shells to meet requirements of the membrane theory include: (*a*) a device to ensure a reaction tangent to the middle surface; (*b*) stiffened edges, such as the ring girder at the base of a dome; (*c*) gradually increased shell thicknesses at a stiffening member; (*d*) a transition curve at changes in section; (*e*) a stiffening edge obtained by thickening the shell; (*f*) scalloped edges; (*g*) a flared support.

transmits normal forces to columns below. The enlarged view of the ring girder in Fig. 5.97*c* shows gradual thickening of the shell to reduce the abruptness of the change in section. The stiffening ring at the lantern in Fig. 5.97*d*, extending around the opening at the crown, projects above the middle surface, for compatibility of strains, and connects through a transition curve with the shell; often, the rim need merely be thickened when the edge is upturned, and the ring can be omitted. In Fig. 5.97*e*, the boundary of the shell is a stiffened edge. In Fig. 5.97*f*, a scalloped shell provides gradual tapering for transmitting the loads to the supports, at the same time providing access to the shell enclosure. And in Fig. 5.97*g*, a column is flared widely at the top to support a thin shell at an interior point.

Even when the conditions for geometric compatibility are not satisfactory, the membrane theory is a useful approximation. Furthermore, it yields a particular solution to the differential equations of the bending theory.

(D. P. Billington, "Thin Shell Concrete Structures," 2d ed., and S. Timoshenko and S. Woinowsky-Krieger, "Theory of Plates and Shells," McGraw-Hill Book Company, New York: V. S. Kelkar and R. T. Sewell, "Fundamentals of the Analysis and Design of Shell Structures," Prentice-Hall, Englewood Cliffs, N.J.)

5.15.3 Bending Theory for Thin Shells

When equilibrium conditions are not satisfied or incompatible deformations exist at boundaries, bending and torsion stresses arise in the shell. Sometimes, the design of the shell and its supports can be modified to reduce or eliminate these stresses (Art. 5.15.2). When the design cannot eliminate them, provisions must be made for the shell to resist them.

But even for the simplest types of shells and loading, the stresses are difficult to compute. In bending theory, a thin shell is statically indeterminate; deformation conditions must supplement equilibrium conditions in setting up differential equations for determining the unknown forces and moments. Solution of the resulting equations may be tedious and time-consuming, if indeed solution if possible.

In practice, therefore, shell design relies heavily on the designer's experience and judgment. The designer should consider the type of shell, material of which it is made, and support and boundary conditions, and then decide whether to apply a bending theory in full, use an approximate bending theory, or make a rough estimate of the effects of bending and torsion. (Note that where the effects of a disturbance are large, these change the normal forces and shears computed by the membrane theory.) For concrete domes, for example, the usual procedure is to use as support a deep, thick girder or a heavily reinforced or prestressed tension ring, and the shell is gradually thickened in the vicinity of this support (Fig. 5.97*c*).

Circular barrel arches, with ratio of radius to distance between supporting arch ribs less than 0.25 may be designed as beams with curved cross section. Secondary stresses, however, must be taken into account. These include stresses due to volume change of rib and shell, rib shortening, unequal settlement of footings, and temperature differentials between surfaces.

Bending theory for cylinders and domes is given in W. Flügge, "Stresses in Shells," Springer-Verlag, New York; D. P. Billington, "Thin Shell Concrete Structures," 2d ed., and S. Timoshenko and S. Woinowsky-Krieger, "Theory of Plates and Shells," McGraw-Hill Book Company, New York; "Design of Cylindrical Concrete Shell Roofs," Manual of Practice No. 31, American Society of Civil Engineers.

5.15.4 Stresses in Thin Shells

The results of the membrane and bending theories are expressed in terms of unit forces and unit moments, acting per unit of length over the thickness of the shell. To compute the unit stresses from these forces and moments, usual practice is to assume normal forces and shears to be uniformly distributed over the shell thickness and bending stresses to be linearly distributed.

Then, normal stresses can be computed from equations of the form

$$f_x = \frac{N_x}{t} + \frac{M_x}{t^3/12} z \tag{5.175}$$

where z = distance from middle surface
t = shell thickness
M_x = unit bending moment about axis parallel to direction of unit normal force N_x

Similarly, shearing stresses produced by central shears and twisting moments may be calculated from equations of the form

$$v_{xy} = \frac{T}{t} \pm \frac{D}{t^3/12} z \tag{5.176}$$

where D = twisting moment and T = unit shear force along the middle surface. Normal shearing stresses may be computed on the assumption of a parabolic stress distribution over the shell thickness:

$$v_{xz} = \frac{V}{t^3/t}\left(\frac{t^2}{4} - z^2\right) \tag{5.177}$$

where V = unit shear force normal to middle surface.

5.15.5 Folded Plates

A folded-plate structure consists of a series of thin planar elements, or flat plates, connected to one another along their edges. Usually used on long spans, especially for roofs, folded plates derive their economy from the girder action of the plates and the mutual support they give one another.

Longitudinally, the plates may be continuous over their supports. Transversely, there may be several plates in each bay (Fig. 5.98). At the edges, or folds, they may be capable of transmitting both moment and shear or only shear.

A folded-plate structure has a two-way action in transmitting loads to its supports. Transversely, the elements act as slabs spanning between plates on either side. The plates then act as girders in carrying the load from the slabs longitudinally to supports, which must be capable of resisting both horizontal and vertical forces.

If the plates are hinged along their edges, the design of the structure is relatively simple. Some simplification also is possible if the plates, though having integral edges, are steeply sloped or if the span is sufficiently long with respect to other dimensions that beam theory applies. But there are no criteria for determining when such simplification is possible with acceptable accuracy. In general, a reasonably accurate analysis of folded-plate stresses is advisable.

Several good methods are available (D. Yitzhaki, "The Design of Prismatic and Cylindrical Shell Roofs," North Holland Publishing Company, Amsterdam; "Phase I Report on Folded-plate Construction," Proceedings Paper 3741, *Journal of the Structural Division, American Society of Civil Engineers,* December 1963; and A. L. Parme and J. A. Sbarounis, "Direct Solution of Folded Plate Concrete Roofs," EB021D, Portland Cement Association, Skokie, Ill.). They all take into account the effects of plate deflections on the slabs and usually make the following assumptions:

The material is elastic, isotropic, and homogeneous. The longitudinal distribution of all loads on all plates is the same. The plates carry loads transversely only by

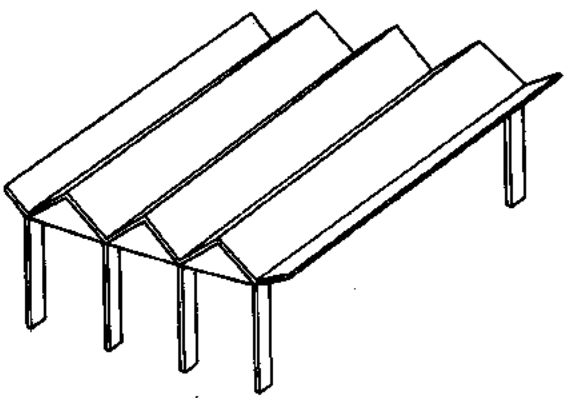


FIGURE 5.98 Folded-plate structure.

bending normal to their planes and longitudinally only by bending within their planes. Longitudinal stresses vary linearly over the depth of each plate. Supporting members, such as diaphragms, frames, and beams, are infinitely stiff in their own planes and completely flexible normal to their own planes. Plates have no torsional stiffness normal to their own planes. Displacements due to forces other than bending moments are negligible.

Regardless of the method selected, the computations are rather involved; so it is wise to carry out the work by computer or, when done manually, in a well-organized table. The Yitzhaki method offers some advantages over others in that the calculations can be tabulated, it is relatively simple, it requires the solution of no more simultaneous equations than one for each edge for simply supported plates, it is flexible, and it can easily be generalized to cover a variety of conditions.

Yitzhaki Method. Based on the assumptions and general procedure given above, the Yitzhaki method deals with the slab and plate systems that comprise a folded-plate structure in two ways. In the first, a unit width of slab is considered continuous over supports immovable in the direction of the load (Fig. 5.99*b*). The strip usually is taken where the longitudinal plate stresses are a maximum. Second, the slab reactions are taken as loads on the plates, which now are assumed to be hinged along the edged (Fig. 5.99*c*). Thus, the slab reactions cause angle changes in the plates at each fold. Continuity is restored by applying to the plates an unknown moment at each edge. The moments can be determined from the fact that at each edge the sum of the angle changes due to the loads and to the unknown moments must equal zero.

The angle changes due to the unknown moments have two components. One is the angle change at each slab end, now hinged to an adjoining slab, in the transverse strip of unit width. The second is the angle change due to deflection of the plates. The method assumes that the angle change at each fold varies in the same way longitudinally as the angle changes along the other folds.

For example, for the folded-plate structure in Fig. 5.99*a*, the steps in analysis are as follows:

Step 1. Compute the loads on a 12-in-wide transverse strip at midspan.

Step 2. Consider the strip as a continuous slab supported at the folds (Fig. 5.99*b*), and compute the bending moments by moment distribution.

Step 3. From the end moments M found in Step 2, compute slab reactions and plate loads. Reactions (positive upward) at the nth edge are

$$R_n = V_n + V_{n+1} + \frac{M_{n-1} + M_n}{a_n} - \frac{M_n + M_{n+1}}{a_{n+1}} \tag{5.178}$$

where V_n, V_{n+1} = shears at both sides of edge n
M_n = moment at edge n
M_{n-1} = moment at edge $(n - 1)$
M_{n+1} = moment at edge $(n + 1)$
a = horizontal projection of depth h

Let $k = \tan \phi_n - \tan \phi_{n+1}$, where ϕ is positive as shown in Fig. 99*a*. Then, the load (positive downward) on the nth plate is

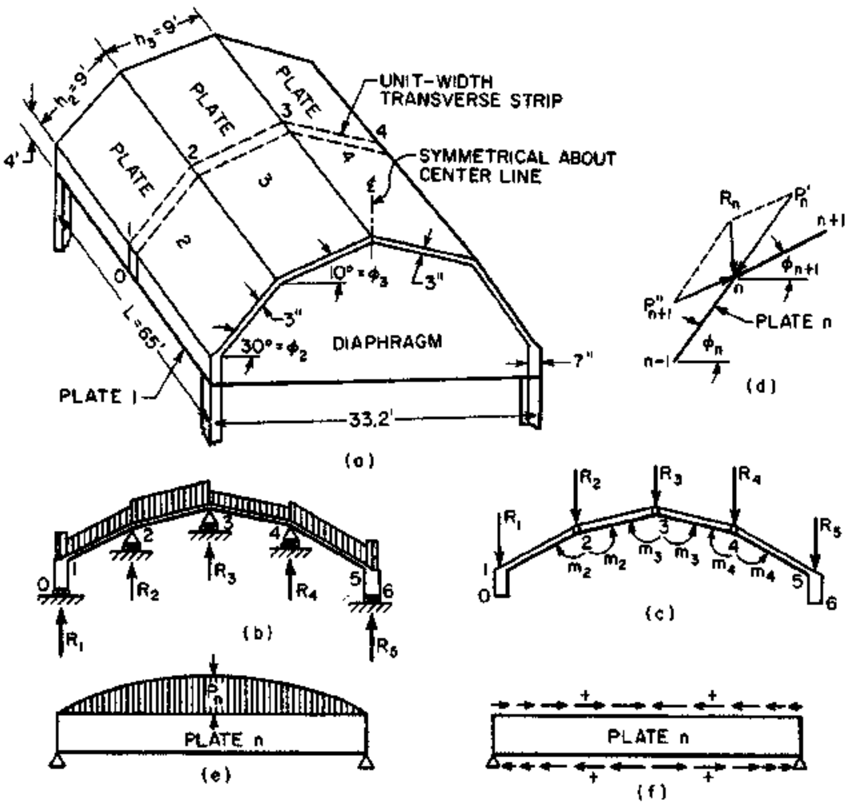


FIGURE 5.99 Folded plate is analyzed by first considering a transverse strip (*a*) as a continuous slab on supports that do not settle (*b*). then, (*c*) the slabs are assumed hinged and acted upon by the reactions computed in the first step and by unknown moments to correct for this assumption. (*d*) Slab reactions are resolved into plate forces, parallel to the planes of the plates. (*e*) In the longitudinal direction, the plates act as deep girders with shears along the edges. (*f*) Arrows indicate the positive directions for the girder shears.

$$P_n = \frac{R_n}{k_n \cos \phi_n} - \frac{R_{n-1}}{k_{n-1} \cos \phi_n} \tag{5.179}$$

(Figure 5.99*d* shows the resolution of forces at edge n; $n - 1$ is similar.) Equation (5.179) does not apply for the case of a vertical reaction on a vertical plate, for R/k is the horizontal component of the reaction.

Step 4. Calculate the midspan (maximum) bending moment in each plate. In this example, each plate is a simple beam and $M = PL^2/8$, where L is the span in feet.

Step 5. Determine the free-edge longitudinal stresses at midspan. In each plate, these can be computed from

$$f_{n-1} = \frac{72M}{Ah} \qquad f_n = -\frac{72M}{Ah} \tag{5.180}$$

where f is the stress in psi, M the moment in ft-lb from Step 4, A = plate cross-sectional area and tension is taken as positive, compression as negative.

Step 6. Apply a shear to adjoining edges to equalize the stresses there. Compute the adjusted stresses by converging approximations, similar to moment distribution. To do this, distribute the unbalanced stress at each edge in proportion to the reciprocals of the areas of the plates, and use a carry-over factor of $-\frac{1}{2}$ to distribute the tress to a far edge. Edge 0, being a free edge, requires no distribution of the stress there. Edge 3, because of symmetry, may be treated the same, and distribution need be carried out only in the left half of the structure.

Step 7. Compute the midspan edge deflections. In general, the vertical component δ can be computed from

$$\frac{E}{L^2}\,\delta_n = \frac{15}{k_n}\left(\frac{f_{n-1} - f_n}{a_n} - \frac{f_n - f_{n+1}}{a_{n+1}}\right) \tag{5.181}$$

where E = modulus of elasticity, psi
$k = \tan\,\phi_n - \tan\,\phi_{n+1}$, as in Step 3

The factor E/L^2 is retained for convenience; it is eliminated by dividing the simultaneous angle equations by it. For a vertical plate, the vertical deflection is given by

$$\frac{E}{L^2}\,\delta_n = \frac{15(f_{n-1} - f_n)}{h_n} \tag{5.182}$$

Step 8. Compute the midspan angle change θ_P at each edge. This can be determined from

$$\frac{E}{L^2}\,\theta_P = -\frac{\delta_{n-1} - \delta_n}{a_n} + \frac{\delta_n - \delta_{n+1}}{a_{n+1}} \tag{5.183}$$

Step 9. To correct the edge rotations with a symmetrical loading, apply an unknown moment of $+100m_n \sin\,(\pi x/L)$, in-lb (positive when clockwise) to plate n at edge n and $-1000m_n \sin\,(\pi x/L)$ to its counterpart, plate n' at edge n'. Also, apply $-1000m_n \sin\,(\pi x/L)$ to plate $(n + 1)$ at edge n and $+1000m_n \sin\,(\pi x/L)$ sine function is assumed to make the loading vary longitudinally in approximately the same manner as the deflections.) At midspan, the absolute value of these moments is $1000m_n$.

The 12-in-wide transverse strip at midspan, hinged at the supports, will then be subjected at the supports to moments of $1000m_n$. Compute the rotations thus caused in the slabs from

$$\frac{E}{L^2}\,\theta''_{n-1} = \frac{166.7h_n m_n}{L^2 t_n^3}$$

$$\frac{E}{L^2}\,\theta''_n = \frac{333.3m_n}{L^2}\left(\frac{h_n}{t_n^3} + \frac{h_{n+1}}{t_{n+1}^3}\right) \tag{5.184}$$

$$\frac{E}{L^2}\,\theta''_{n+1} = \frac{166.7h_{n+1}m_n}{L^2 t_{n+1}^3}$$

Step 10. Compute the slab reactions and plate loads due to the unknown moments. The reactions are

$$R_{n-1} = \frac{1000m_n}{a_n} \quad R_n = 1000m_n\left(\frac{1}{a_n} + \frac{1}{a_{n+1}}\right) \quad R_{n+1} = -\frac{1000m_n}{a_{n+1}} \tag{5.185}$$

The plate loads are

$$P_n = \frac{1}{\cos\phi_n}\left(\frac{R_n}{k_n} - \frac{R_{n-1}}{k_{n-1}}\right) \tag{5.186}$$

Step 11. Assume that the loading on each plate is $P_n \sin(\pi x/L)$ (Fig. 5.99*e*), and calculate the midspan (maximum) bending moment. For a simple beam,

$$M = \frac{PL^2}{\pi^2}$$

Step 12. Using Eq. (5.180), compute the free-edge longitudinal stresses at midspan. Then, as in Step 6, apply a shear at each edge to equalize the stresses. Determine the adjusted stresses by converging approximations.

Step 13. Compute the vertical component of the edge deflections at midspan from

$$\frac{E}{L^2}\,\delta_n = \frac{144}{\pi^2 k_n}\left(\frac{f^{n-1} - f_n}{a_n} - \frac{f_n - f_{n+1}}{a_{n+1}}\right) \tag{5.187}$$

or for a vertical plate from

$$\frac{E}{L^2}\,\delta_n = \frac{144(f_{n-1} - f_n)}{\pi^2 h_n} \tag{5.188}$$

Step 14. Using Eq. (5.183), determine the midspan angle change θ' at each edge.

Step 15. At each edge, set up an equation by putting the sum of the angle changes equal to zero. Thus, after division by E/L^2: $\theta_P + \theta'' + \Sigma\theta' = 0$. Solve these simultaneous equations for the unknown moments.

Step 16. Determine the actual reactions, loads, stresses, and deflections by substituting for the moments the values just found.

Step 17. Compute the shear stresses. The shear stresses at edge n (Fig. 5.99*f*) is

$$T_n = T_{n-1} \frac{f_{n-1} + f_n}{2} A_n \tag{5.189}$$

In the example, $T_o = 0$, so the shears at the edges can be obtained successively, since the stresses f are known.

For a uniformly loaded folded plate, the shear stress S, psi, at any point on an edge n is approximately

$$S = \frac{2T_{\max}}{3Lt} \left(\frac{1}{2} - \frac{x}{L} \right) \tag{5.190}$$

With a maximum at plate ends of

$$S_{\max} = \frac{T_{\max}}{3Lt} \tag{5.191}$$

The shear stress, psi, at middepth (not always a maximum) is

$$v_n = \left(\frac{3P_nL}{2A_n} + \frac{S_{n-1} + S_n}{2} \right) \left(\frac{1}{2} - \frac{x}{L} \right) \tag{5.192}$$

and has its largest value at $x = 0$:

$$v_{\max} = \frac{0.75P_nL}{A_n} + \frac{S_{n-1} + S_n}{4} \tag{5.193}$$

For more details, see D. Yitzhaki and Max Reiss, "Analysis of Folded Plates," Proceedings Paper 3303, *Journal of the Structural Division, American Society of Civil Engineers,* October 1962.

5.16 CABLE-SUPPORTED STRUCTURES*

A cable is a linear structural member, like a bar of a truss. The cross-sectional dimensions of a cable relative to its length, however, are so small that it cannot withstand bending or compression. Consequently, under loads at an angle to its longitudinal axis, a cable sags and assumes a shape that enables it to develop tensile stresses that resist the loads.

Structural efficiency results from two cable characteristics: (1) uniformity of tensile stresses over the cable cross section, and (2) usually, small variation of tension along the longitudinal axis. Hence, it is economical to use materials with very high tensile strength for cables.

Cables sometimes are used in building construction as an alternative to such tension members as hangers, ties, or tension chords of trusses. For example, cables are used in a form of long-span cantilever-truss construction in which a horizontal

*Reprinted with permission from F. S. Merritt, "Structural Steel Designers' Handboo," McGraw-Hill Book Company, New York.

roof girder is supported at one end by a column and near the other end by a cable that extends diagonally upward to the top of a vertical mast above the column support (cable-stayed-girder construction, Fig. 5.100). Cable stress an be computed for this case from the laws of equilibrium.

Cables also may be used in building construction instead of girders, trusses, or membranes to support roofs, For the purpose, cables may be arranged in numerous ways. It is consequently impractical to treat in detail in this book any but the simplest types of such applications of cables. Instead, general procedures for analyzing cable-supported structures are presented in the following.

5.16.1 Simple Cables

An ideal cable has o resistance to bending. Thus, in analysis of a cable in equilibrium, not only is the sum of the moments about any point equal to zero but so is the bending moment at any point. Consequently, the equilibrium shape of the cable corresponds to the funicular, or bending-moment, diagram for the loading (Fig. 5.101*a*). As a result, the tensile force at any point of the cable is tangent there to the cable curve.

The point of maximum sag of a cable coincides with the point of zero shear. (Sag in this case should be measured parallel to the direction of the shear forces.)

Stresses in a cable are a function of the deformed shape. Equations needed for analysis, therefore, usually are nonlinear. Also, in general, stresses and deformations cannot be obtained accurately by superimposition of loads. A common procedure

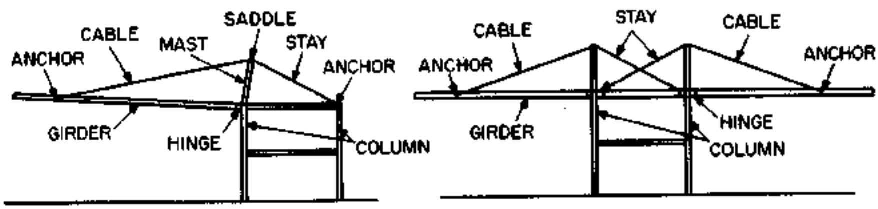


FIGURE 5.100 Two types of cable-stayed girder construction for roofs.

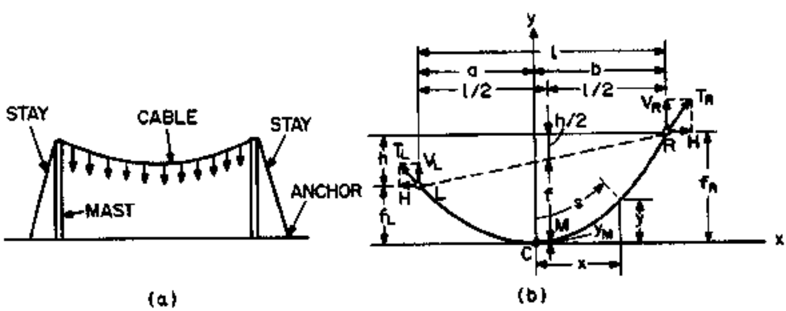


FIGURE 5.101 Simple cable: (*a*) cable with a uniformly distributed load; (*b*) cable with supports at different levels.

in analysis is to obtain a solution in steps by using linear equations to approximate the nonlinear ones and by starting with the initial geometry to obtain better estimates of the final geometry.

For convenience in analysis, the cable tension, directed along the cable curve, usually is resolved into two components. Often, it is advantageous to resolve the tension T into a horizontal component H and a vertical component V (Fig. 5.100*b*). Under vertical loading then, the horizontal component is constant along the cable. Maximum tension occurs at the support. V is zero at the point of maximum sag.

For a general, distributed vertical load q, the cable must satisfy the second-order linear differential equation

$$Hy^n = q \tag{5.194}$$

where y = rise of cable at distance x from low point (Fig. 5.100*b*)
$y^n = d^2y/dx^2$

Catenary. Weight of a cable of constant cross-section represents a vertical loading that is uniformly distributed along the length of cable. Under such a loading, a cable takes the shape of a catenary.

Take the origin of coordinates at the low point C and measure distance s along the cable from C (Fig. 5.100*b*). If q_o is the load per unit length of cable, Eq. (5.194) becomes

$$Hy^n = \frac{q_o\,ds}{dx} = q_o\sqrt{1 + y'^2} \tag{5.195}$$

where $y' = dy/dx$. Solving for y' gives the slope at any point of the cable

$$y' = \sinh\frac{q_o x}{H} = \frac{q_o x}{H} + \frac{1}{3!}\left(\frac{q_o x}{H}\right)^3 + \cdots \tag{5.196}$$

A second integration then yields the equation for the cable shape, which is called a catenary.

$$y = \frac{H}{q_o}\left(\cosh\frac{q_o x}{H} - 1\right) = \frac{q_o}{H}\frac{x^2}{2!} + \left(\frac{q_o}{H}\right)^3\frac{x^4}{4!} + \cdots \tag{5.197}$$

If only the first term of the series expansion is used, the cable equation represents a parabola. Because the parabolic equation usually is easier to handle, a catenary often is approximated by a parabola.

For a catenary, length of arc measured from the low point is

$$s = \frac{H}{q_o}\sinh\frac{q_o x}{H} = x + \frac{1}{3!}\left(\frac{q_o}{H}\right)^2 x^3 + \cdots \tag{5.198}$$

Tension at any point is

$$T = \sqrt{H^2 + q_o^2 s^2} = H + q_o y \tag{5.199}$$

The distance from the low point C to the left support L is

$$a = \frac{H}{q_o} \cosh^{-1} \left(\frac{q_o}{H} f_L + 1 \right) \tag{5.200}$$

where f_L = vertical distance from C to L. The distance from C to the right support R is

$$b = \frac{H}{q_o} \cosh^{-1} \left(\frac{q_o}{H} f_R + 1 \right) \tag{5.201}$$

where f_R = vertical distance from C to R.

Given the sags of a catenary f_L and f_R under a distributed vertical load q_o, the horizontal component of cable tension H may be computed from

$$\frac{q_o l}{H} = \cosh^{-1} \left(\frac{q_o f_L}{H} + 1 \right) + \cosh^{-1} \left(\frac{q_o f_R}{H} + 1 \right) \tag{5.202}$$

where l = span, or horizontal distance between supports L and $R = a + b$. This equation usually is solved by trial. A first estimate of H for substitution in the right-hand side of the equation may be obtained by approximating the catenary by a parabola. Vertical components of the reactions at the supports can be computed from

$$R_L = H \sinh \frac{q_o a}{H} \qquad R_R = H \sinh \frac{q_o b}{H} \tag{5.203}$$

Parabola. Uniform vertical live loads and uniform vertical dead loads other than cable weight generally may be treated as distributed uniformly over the horizontal projection of the cable. Under such loadings, a cable takes the shape of a parabola.

Take the origin of coordinates at the low point C (Fig. 5.100*b*). If w_o is the load per foot horizontally, Eq. (5.194) becomes

$$Hy^n = w_o \tag{5.204}$$

Integration gives the slope at any point of the cable

$$y' = \frac{w_o x}{H} \tag{5.205}$$

A second integration yields the parabolic equation for the cable shape

$$y = \frac{w_o x^2}{2H} \tag{5.206}$$

The distance from the low point C to the left support L is

$$a = \frac{l}{2} - \frac{Hh}{w_o l} \tag{5.207}$$

where l = span, or horizontal distance between supports L and $R = a + b$
h = vertical distance between supports

The distance from the low point C to the right support R is

$$b = \frac{l}{2} + \frac{Hh}{w_o l} \tag{5.208}$$

When supports are not at the same level, the horizontal component of cable tension H may be computed from

$$H = \frac{w_o l^2}{h^2}\left(f_R - \frac{h}{2} \pm \sqrt{f_L f_R}\right) = \frac{w_o l^2}{8f} \tag{5.209}$$

where f_L = vertical distance from C to L
f_R = vertical distance from C to R
f = sag of cable measured vertically from chord LR midway between supports (at $x = Hh/w_o l$)

As indicated in Fig. 5.100*b*,

$$f = f_L + \frac{h}{2} - y_M \tag{5.210}$$

where $y_M = Hh^2/2w_o l^2$. The minus sign should be used in Eq. (5.209) when low point C is between supports. If the vertex of the parabola is not between L and R, the plus sign should be used.

The vertical components of the reactions at the supports can be computed from

$$V_L = w_o a = \frac{w_o l}{2} - \frac{Hh}{l} \qquad V_R = w_o b = \frac{w_o l}{2} + \frac{Hh}{l} \tag{5.211}$$

Tension at any point is

$$T = \sqrt{H^2 + w_o^2 x^2} \tag{5.212}$$

Length of parabolic arc RC is

$$L_{RC} = \frac{b}{2}\sqrt{1 + \left(\frac{w_o b}{KH}\right)^2} + \frac{H}{2w_o}\sinh\frac{w_o b}{H} = b + \frac{1}{6}\left(\frac{w_o}{H}\right)^2 b^3 + \cdots \tag{5.213}$$

Length of parabolic are LC is

$$L_{LC} = \frac{a}{2}\sqrt{1 + \left(\frac{w_o a}{H}\right)^2} + \frac{H}{2w_o}\sinh\frac{w_o a}{H} = a + \frac{1}{6}\left(\frac{w_o}{H}\right)^2 a^3 + \cdots \tag{5.214}$$

When supports are at the same level, $f_L = f_R = f$, $h = 0$, and $a = b = l/2$. The horizontal component of cable tension H may be computed from

$$H = \frac{w_o l^2}{8f} \tag{5.215}$$

The vertical components of the reactions at the supports are

$$V_L = V_R = \frac{w_o l}{2} \tag{5.216}$$

Maximum tension occurs at the supports and equals

$$T_L = T_R = \frac{w_o l}{2} \sqrt{1 + \frac{l^2}{16f^2}} \tag{5.217}$$

Length of cable between supports is

$$\begin{aligned} L &= \frac{1}{2} \sqrt{1 + \left(\frac{w_o l}{2H}\right)^2} + \frac{H}{w_o} \sinh \frac{w_o l}{2H} \\ &= l \left(1 + \frac{8}{3}\frac{f^2}{l^2} - \frac{32}{5}\frac{f^4}{l^4} + \frac{256}{7}\frac{f^6}{l^6} + \cdots\right) \end{aligned} \tag{5.218}$$

If additional uniformly distributed load is applied to a parabolic cable, the change in sag is approximately

$$\Delta f = \frac{15}{16}\frac{l}{f}\frac{\Delta L}{5 - 24f^2/l^2} \tag{5.219}$$

For a rise in temperature t, the change in sag is about

$$\Delta f = \frac{15}{16}\frac{l^2 ct}{f(5 - 24f^2/l^2)}\left(1 + \frac{8}{3}\frac{f^2}{l^2}\right) \tag{5.220}$$

where c = coefficient of thermal expansion.

Elastic elongation of a parabolic cable is approximately

$$\Delta L = \frac{Hl}{ARE}\left(1 + \frac{16}{3}\frac{f^2}{l^2}\right) \tag{5.221}$$

where A = cross-sectional area of cable
E = modulus of elasticity of cable steel
H = horizontal component of tension in cable

If the corresponding change in sag is small, so that the effect on H is negligible, this change may be computed from

$$\Delta f = \frac{15}{16}\frac{Hl^2}{AREf}\frac{1 + 16f^2/3l^2}{5 - 24f^2/l^2} \tag{5.222}$$

For the general case of vertical dead load on a cable, the initial shape of the cable is given by

$$y_D = \frac{M_D}{H_D} \tag{5.223}$$

where M_D = dead-load bending moment that would be produced by the load in a simple beam
H_D = horizontal component of tension due to dead load

For the general case of vertical live load on the cable, the final shape of the cable is given by

$$y_D + \delta = \frac{M_D + M_L}{H_D + H_L} \tag{5.224}$$

where δ = vertical deflection of cable due to live load
M_L = live-load bending moment that would be produced by the live load in a simple beam
H_L = increment in horizontal component of tension due to live load

Subtraction of Eq. (5.223) from Eq. (5.224) yield

$$\delta = \frac{M_L - H_L y_D}{H_D + H_L} \tag{5.225}$$

If the cable is assumed to take a parabolic shape, a close approximation to H_L may be obtained from

$$\frac{H_L}{AE} K = \frac{w_D}{H_D} \int_0^l \delta\, dx - \frac{1}{2} \int_0^l \delta'' \delta\, dx \tag{5.226}$$

$$K = l \left[\frac{1}{4} \left(\frac{5}{2} + \frac{16f^2}{l^2} \right) \sqrt{1 + \frac{16f^2}{l^2}} + \frac{3l}{32f} \log_e \left(\frac{4f}{l} + \sqrt{1 + \frac{16f^2}{l^2}} \right) \right] \tag{5.227}$$

where $\delta'' = d^2\delta/dx^2$.

If elastic elongation and δ'' can be ignored, Eq. (5.226) simplifies to

$$H_L = \frac{\int_0^l M_L\, dx}{\int_0^l y_D\, dx} = \frac{3}{2fl} \int_0^l M_L\, dx \tag{5.228}$$

Thus, for a load uniformly distributed horizontally w_L,

$$\int_0^l M_L\, dx = \frac{w_L l^3}{12} \tag{5.229}$$

and the increase in the horizontal component of tension due to live load is

$$H_L = \frac{3}{2fl} \frac{w_L l^3}{12} = \frac{w_L l^2}{8f} = \frac{w_L l^2}{8} \frac{8H_D}{w_D l^2} = \frac{w_L}{w_D} H_D \tag{5.230}$$

When a more accurate solution is desired, the value of H_L obtained from Eq. (5.230) can be used for an initial trial in solving Eqs. (5.225) and (5.226).

(S. P. Timoshenko and D. H. Young, "Theory of Structures," McGraw-Hill Book Company, New York: W. T. O'Brien and A. J. Francis, "Cable Movements under Two-dimensional Loads," *Journal of the Structural Division, ASCE,* Vol. 90, No. ST3, *Proceedings Paper* 3929, June 1964, pp. 89–123; W. T. O'Brien, "General Solution of Suspended Cable Problems," *Journal of the Structural Division, ASCE,* Vol. 93, No. ST1, *Proceedings Paper* 5085, February, 1967, pp. 1–26; W. T. O'Brien, "Behavior of Loaded Cable Systems," *Journal of the Structural Division, ASCE,* Vol. 94, No. ST10, *Proceedings Paper* 6162, October 1968, pp. 2281–2302; G. R. Buchanan, "Two-dimensional Cable Analysis," *Journal of the Structural Division, ASCE,* Vol. 96, No. ST7, *Proceedings Paper* 7436, July 1970, pp. 1581–1587).

5.16.2 Cable Systems

Analysis of simple cables is described in Art. 5.16.1. Cables, however, may be assembled into many types of systems. One important reason for such systems is that roofs to be supported are two- or three-dimensional. Consequently, three-dimensional cable arrangements often are advantageous. Another important reason is that cable systems can be designed to offer much higher resistance to vibrations than simple cables do.

Like simple cables, cable systems behave nonlinearly. Thus, accurate analysis is difficult, tedious, and time-consuming. As a result, many designers use approximate methods that appear to have successfully withstood the test of time. Because of the numerous types of systems and the complexity of analysis, only general procedures will be outlined in this article.

Cable systems may be stiffened or unstiffened. Stiffened systems, usually used for suspension bridges are rarely used in buildings. This article will deal only with unstiffened systems, that is, systems where loads are carried to supports only by cables.

Often, unstiffened systems may be classified as a network or as a cable truss, or double-layered plane system.

Networks consist of two or three sets of cables intersecting at an angle (Fig. 5.102). The cables are fastened together at their intersections.

Cable trusses consist of pairs of cables, generally in a vertical plane. One cable of each pair is concave downward, the other concave upward (Fig. 5.103).

Cable Trusses. Both cables of a cable truss are initially tensioned, or prestressed, to a predetermined shape, usually parabolic. The prestress is made large enough that any compression that may be induced in a cable by loads only reduces the tension in the cable; thus, compressive stresses cannot occur. The relative vertical position of the cables is maintained by verticals, or spreaders, or by diagonals. Diagonals in the truss plane do not appear to increase significantly the stiffness of a cable truss.

Figure 5.103 shows four different arrangements of the cables, with spreaders, in a cable truss. The intersecting types (Fig. 5.103*b* and *c*) usually are stiffer than the others, for given size cables and given sag and rise.

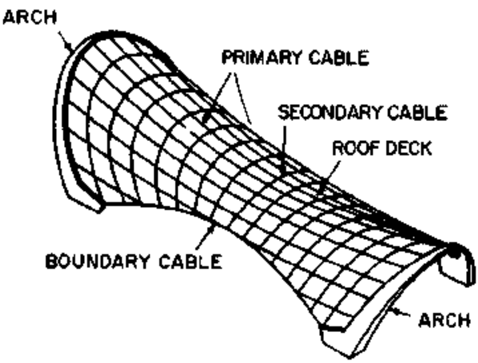


FIGURE 5.102 Cable network.

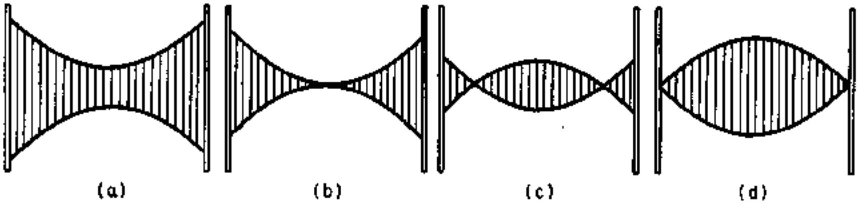


FIGURE 5.103 Planar cable systems: (*a*) completely separated cables; (*b*) cables intersecting at midspan; (*c*) crossing cables; (*d*) cables meeting at supports.

For supporting roofs, cable trusses often are placed radially at regular intervals (Fig. 5.104). Around the perimeter of the roof, the horizontal component of the tension usually is resisted by a circular or elliptical compression ring. To avoid a joint with a jumble of cables at the center, the cables usually are also connected to a tension ring circumscribing the center.

Properly prestressed, such double-layer cable systems offer high resistance to vibrations. Wind or other dynamic forces difficult or impossible to anticipate may cause resonance to occur in a single cable, unless damping is provided. The probability of resonance occurring may be reduced by increasing the dead load on a single cable. But this is not economical, because the size of cable and supports usually must be increased as well. Besides, the tactic may not succeed, because future loads may be outside the design range. Damping, however, may be achieved economically with interconnected cables under different tensions, for example, with cable trusses or networks.

The cable that is concave downward (Fig. 5.103) usually is considered the load-carrying cable. If the prestress in that cable exceeds that in the other cable, the

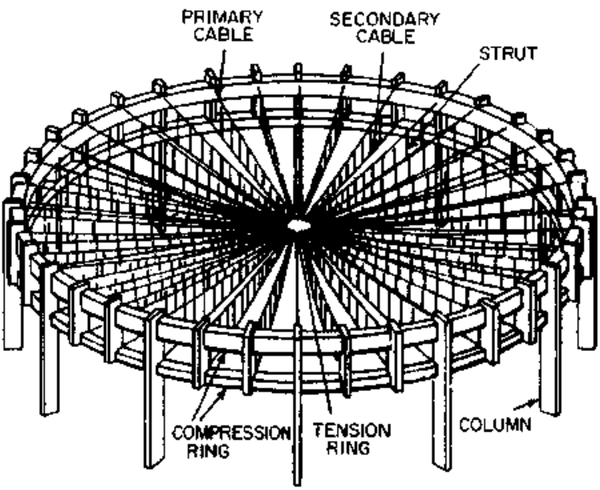


FIGURE 5.104 Cable trusses placed radially to support a round roof.

natural frequencies of vibration of both cables will always differ for any value of live load. To avoid resonance, the difference between the frequencies of the cables should increase with increase in load. Thus, the two cables will tend to assume different shapes under specific dynamic loads. As a consequence, the resulting flow of energy from one cable to the other will dampen the vibrations of both cables.

Natural frequency, cycles per second, of each cable may be estimated from

$$w_n = \frac{n\pi}{l}\sqrt{\frac{Tg}{w}} \tag{5.231}$$

where n = integer, 1 for the fundamental mode of vibration, 2 for the second mode, . . .
l = span of cable, ft
w = load on cable, kips per ft
g = acceleration due to gravity = 32.2 ft/s^2
T = cable tension, kips

The spreaders of a cable truss impose the condition that under a given load the change in sag of the cables must be equal. But the changes in tension of the two cables may not be equal. If the ratio of sag to span f/l is small (less than about 0.1). Eq. (5.222) indicates that, for a parabolic cable, the change in tension is given approximately by

$$\Delta H = \frac{16}{3}\frac{AEf}{l^2}\Delta f \tag{5.232}$$

where Δf = change in sag
A = cross-sectional area of cable
E = modulus of elasticity of cable steel

Double cables interconnected with struts may be analyzed as discrete or continuous systems. For a discrete system, the spreaders are treated as individual members. For a continuous system, the spreaders are replaced by a continuous diaphragm that ensures that the changes in sag and rise of cables remain equal under changes in load. Similarly, for analysis of a cable network, the cables, when treated as a continuous system, may be replaced by a continuous membrane.

(C. H. Mollman, "Analysis of Plane Prestressed Cable Structures," *Journal of the Structural Division, ASCE,* Vol. 96, No. ST10, *Proceedings Paper* 7598, October 1970, pp. 2059–2082; D. P. Greenberg, "Inelastic Analysis of Suspension Roof Structures," *Journal of the Structural Division, ASCE,* Vol. 96, No. ST5, *Proceedings Paper* 7284, May 1970, pp. 905–930; H. Tottenham and P. G. Williams, "Cable Net: Continuous System Analysis," *Journal of the Engineering Mechanics Division, ASCE,* Vol. 96, No. EM3, *Proceedings Paper* 7347, June 1970, pp. 277–293; A. Siev, "A General Analysis of Prestressed Nets," *Publications, International Association for Bridge and Structural Engineering,* Vol. 23, pp. 283–292, Zurich, Switzerland, 1963; A. Siev, "Stress Analysis of Prestressed Suspended Roofs," *Journal of the Structural Division, ASCE,* Vol. 90, No. ST4, *Proceedings Paper* 4008. August 1964, pp. 103–121; C. H. Thornton and C. Birnstiel, "Three-dimensional Suspension Structures," *Journal of the Structural Division, ASCE,* Vol. 93, No. ST2, *Proceedings Paper* 5196, April 1967, pp. 247–270.)

5.17 AIR-STABILIZED STRUCTURES

A true membrane is able to withstand tension but is completely unable to resist bending. Although it is highly efficient structurally, like a shell, a membrane must be much thinner than a shell and therefore can be made of a very lightweight material, such as fabric, with considerable reduction in dead load compared with other types of construction. Such a thin material, however, would buckle if subjected to compression. Consequently, a true membrane, when loaded, deflects and assumes a shape that enables it to develop tensile stresses that resist the loads.

Membranes may be used for the roof of a building or as a complete exterior enclosure. One way to utilize a membrane for these purposes is to hang it with initial tension between appropriate supports. For example, a tent may be formed by supporting fabric atop one or more tall posts and anchoring the outer edges of the stretched fabric to the ground. As another example, a dish-shaped roof may be constructed by stretching a membrane and anchoring it to the inner surface of a ring girder. In both examples, loads induce only tensile stresses in the membrane. The stresses may be computed from the laws of equilibrium, because a membrane is statically determinate.

Another way to utilize a membrane as an enclosure or roof is to pretension the membrane to enable it to carry compressive loads. For the purpose, forces may be applied, and retained as long as needed, around the edges or over the surface of the membrane to induce tensile stresses that are larger than the larger compressive stresses that loads will impose. As a result, compression due to loads will only reduce the prestress and the membrane will always be subjected only to tensile stresses.

5.17.1 Pneumatic Construction

A common method of pretensioning a membrane enclosure is to pressurize the interior with air. Sufficient pressure is applied to counteract dead loads, so that the membrane actually floats in space. Slight additional pressurization is also used to offset wind and other anticipated loads. Made of lightweight materials, a membrane thus can span large distances economically. This type of construction, however, has the disadvantage that energy is continuously required for operation of air compressors to maintain interior air at a higher pressure than that outdoors.

Pressure differentials used in practice are not large. They often range between 0.02 and 0.04 psi (3 and 5 psf). Air must be continually supplied, because of leakage. While there may be some leakage of air through the membrane, more important sources of air loss are the entrances and exits to the structure. Air locks and revolving doors, however, can reduce these losses.

An air-stabilized enclosure, in effect is a membrane bag held in place by small pressure differentials applied by environmental energy. Such a structure is analogous to a soap film. The shape of a bubble is determined by surface-tension forces. The membrane is stressed equally in all directions at every point. Consequently, the film forms shapes with minimum surface area, frequently spherical. Because of the stress distribution, any shape that can be obtained with soap films is feasible for an air-stabilized enclosure. Figure 5.105*c* shows a configuration formed by a conglomeration of bubbles as an illustration of a shape that can be adopted for an air-stabilized structure.

In practice, shapes of air-stabilized structures often resemble those used for thin-shell enclosures. For example, spherical domes (Fig. 5.105*a*) are frequently con-

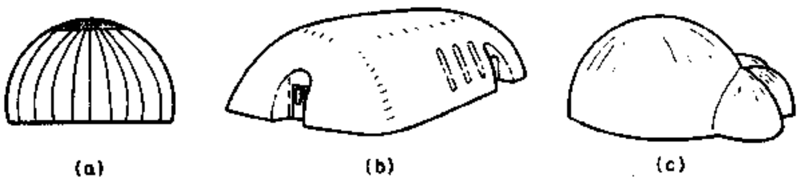


FIGURE 5.105 Some shapes for air-supported structures. (*Reprinted with permission from F. S. Merritt, "Building Engineering and Systems Design," Van Nostrand Reinhold Company, New York.*)

structed with a membrane. Also, membranes are sometimes shaped as semi-circular cylinders with quarter-sphere ends (Fig. 5.105*b*).

Air-stabilized enclosures may be classified as air-inflated, air-supported, or hybrid structures, depending on the type of support.

Air-inflated enclosures are completely supported by pressurized air entrapped within membranes. There are two main types, inflated-rib structures and inflated dual-wall structures.

In inflated-rib construction, the membrane enclosure is supported by a framework of air-pressurized tubes, which serve much like arch ribs in thin-shell construction (Art. 5.15.1). The principle of their action is demonstrated by a water hose. A flexible hose, when empty, collapses under its own weight on short spans or under loads normal to its length; but it stiffens when filled with water. The water pressure tensions the hose walls and enables them to withstand compressive stresses.

In inflated dual-walled construction, pressurized air is trapped between two concentric membranes (Fig. 5.106). The shape of the inner membrane is maintained by suspending it from the outer one. Because of the large volume of air compressed between the membranes, this type of construction can span longer distances than can inflated-rib structures.

Because of the variation of air pressure with changes in temperature, provision must be made for adjustment of the pressure of the compressed air in air-inflated structures. Air must be vented to relieve excessive pressures, to prevent overtensioning of the membranes. Also, air must be added to compensate for pressure drops, to prevent collapse.

Air-supported enclosures consist of a single membrane supported by the difference between internal air pressure and external atmospheric pressure (Fig. 5.107). The pressure differential deflects the membrane outward, inducing tensile stresses in it, thus enabling it to withstand compressive forces. To resist the uplift, the construction must be securely anchored to the ground. Also, the membrane must be completely sealed around its perimeter to prevent air leakage.

Hybrid structures consist of one of the preceding types of pneumatic construction augmented by light metal framing, such as cables. The framing may be merely

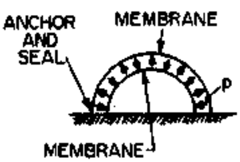


FIGURE 5.106 Inflated dual-wall structure.

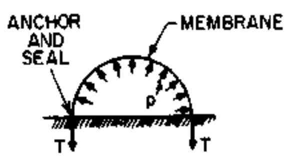


FIGURE 5.107 Air-supported structure.

a safety measure to support the membrane if pressure should be lost or a means of shaping the membrane when it is stretched. Under normal conditions, air pressure against the membrane reduces the load on the framing from heavy wind and snow loads.

5.17.2 Membrane Stresses

Air-supported structures are generally spherical or cylindrical because of the supporting uniform pressure.

When a spherical membrane with radius R, in, its subjected to a uniform radial internal pressure, p, psi, the internal unit tensile force, lb/in, in any direction, is given by

$$T = \frac{pR}{2} \tag{5.233}$$

In a cylindrical membrane, the internal unit tensile force, lb/in, in the circumferential direction is given by

$$T = pR \tag{5.234}$$

where R = radius, in, of the cylinder. The longitudinal membrane stress depends on the conditions at the cylinder ends. For example, with immovable end enclosures, the longitudinal stress would be small. If, however the end enclosure is flexible, a tension about half that given by Eq. (5.234) would be imposed on the membrane in the longitudinal direction.

Unit stress in the membrane can be computed by dividing the unit force by the thickness, in, of the membrane.

(R. N. Dent, "Principles of Pneumatic Architecture," John Wiley & Sons, Inc., New York; J. W. Leonard, "Tension Structures," McGraw-Hill Publishing Company, New York.)

5.18 STRUCTURAL DYNAMICS

Article 5.1.1 notes that loads can be classified as static or dynamic and that the distinguishing characteristic is the rate of application of load. If a load is applied slowly, it may be considered static. Since dynamic loads may produce stresses and deformations considerably larger than those caused by static loads of the same magnitude, it is important to know reasonably accurately what is meant by slowly.

A useful definition can be given in terms of the natural period of vibration of the structure or member to which the load is applied. If the time in which a load rises from zero to its maximum value is more than double the natural period, the load may be treated as static. Loads applied more rapidly may be dynamic. Structural analysis and design for such loads are considerably different from and more complex than those for static loads.

In general, exact dynamic analysis is possible only for relatively simple structures, and only when both the variation of load and resistance with time are a convenient mathematical function. Therefore, in practice, adoption of approximate methods that permit rapid analysis and design is advisable. And usually, because of uncertainties in loads and structural resistance, computations need not be carried out with more than a few significant figures, to be consistent with known conditions.

5.18.1 Properties of Materials under Dynamic Loading

In general mechanical properties of structural materials improve with increasing rate of load application. For low-carbon steel, for example, yield strength, ultimate strength, and ductility rise with increasing rate of strain. Modulus of elasticity in the elastic range, however, is unchanged. For concrete, the dynamic ultimate strength in compression may be much greater than the static strength.

Since the improvement depends on the material and the rate of strain, values to use in dynamic analysis and design should be determined by tests approximating the loading conditions anticipated.

Under many repetitions of loading, though, a member or connection between members may fail because of ''fatigue'' at a stress smaller than the yield point of the material. In general, there is little apparent deformation at the start of a fatigue failure. A crack forms at a point of high stress concentration. As the stress is repeated, the crack slowly spreads, until the member ruptures without measurable yielding. Though the material may be ductile, the fracture looks brittle.

Some materials (generally those with a well-defined yield point) have what is known as an **endurance limit.** This is the maximum unit stress that can be repeated, through a definite range, an indefinite number of times without causing structural damage. Generally, when no range is specified, the endurance limit is intended for a cycle in which the stress is varied between tension and compression stresses of equal value.

A range of stress may be resolved into two components—a steady, or mean, stress and an alternating stress. The endurance limit sometimes is defined as the maximum value of the alternating stress that can be superimposed on the steady stress an indefinitely large number of times without causing fracture.

Design of members to resist repeated loading cannot be executed with the certainty with which members can be designed to resist static loading. Stress concentrations may be present for a wide variety of reasons, and it is not practicable to calculate their intensities. But sometimes it is possible to improve the fatigue strength of a material or to reduce the magnitude of a stress concentration below the minimum value that will cause fatigue failure.

In general, avoid design details that cause severe stress concentrations or poor stress distribution. Provide gradual changes in section. Eliminate sharp corners and notches. Do not use details that create high localized constraint. Locate unavoidable stress raisers at points where fatigue conditions are the least severe. Place connections at points where stress is low and fatigue conditions are not severe. Provide structures with multiple load paths or redundant members, so that a fatigue crack in any one of the several primary members is not likely to cause collapse of the entire structure.

Fatigue strength of a material may be improved by cold-working the material in the region of stress concentration, by thermal processes, or by prestressing it in such a way as to introduce favorable internal stresses. Where fatigue stresses are unusually severe, special materials may have to be selected with high energy absorption and notch toughness.

(J. H. Faupel, ''Engineering Design,'' John Wiley & Sons, Inc., New York; C. H. Norris et al., ''Structural Design for Dynamic Loads,'' McGraw-Hill Book

Company, New York; W. H. Munse, "Fatigue of Welded Steel Structures," Welding Research Council, 345 East 47th Street, New York, NY 10017.)

5.18.2 Natural Period of Vibration

A preliminary step in dynamic analysis and design is determination of this period. It can be computed in many ways, including by application of the laws of conservation of energy and momentum or Newton's second law, $F = M(dv/dt)$, where F is force, M mass, v velocity, and t time. But in general, an exact solution is possible only for simple structures. Therefore, it is general practice to seek an approximate—but not necessarily inexact—solution by analyzing an idealized representation of the actual member or structure. Setting up this model and interpreting the solution require judgment of a high order.

Natural period of vibration is the time required for a structure to go through one cycle of free vibration, that is, vibration after the disturbance causing the motion has ceased.

To compute the natural period, the actual structure may be conveniently represented by a system of masses and massless springs, with additional resistances provided to account for energy losses due to friction, hysteresis, and other forms of damping. In simple cases, the masses may be set equal to the actual masses; otherwise, equivalent masses may have to be computed (Art. 5.18.6). The spring constants are the ratios of forces to deflections.

For example, a single mass on a spring (Fig. 5.108*b*) may represent a simply supported beam with mass that may be considered negligible compared with the load W at midspan (Fig. 5.108*a*). The spring constant k should be set equal to the

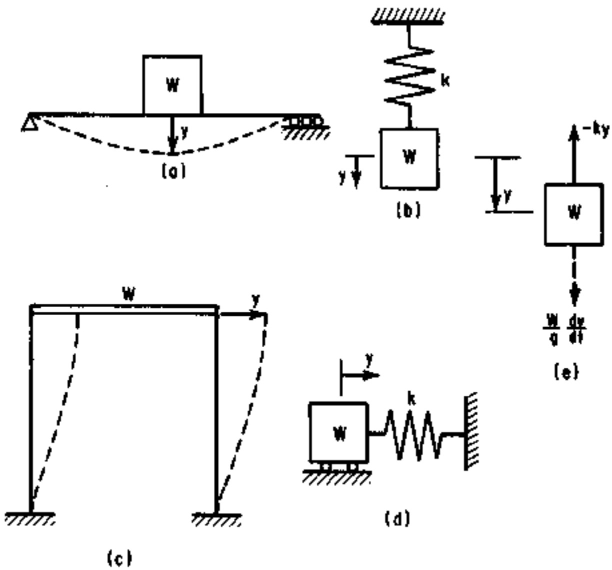


FIGURE 5.108 Mass on a weightless spring (*b*) or (*d*) may represent the motion of (*a*) a beam or (*c*) a rigid frame in free vibration.

load that produces a unit deflection at midspan; thus, $k = 48EI/L^3$, where E is the modulus of elasticity, psi; I the moment of inertia, in^4; and L the span, in, of the beam. The idealized mass equals W/g, where W is the weight of the load, lb, and g is the acceleration due to gravity, 386 in/s^2.

Also, a single mass on a spring (Fig. 5.108*d*) may represent the rigid frame in Fig. 5.108*c*. In that case, $k = 2 \times 12EI/h^3$, where I is the moment of inertia, in^4, of each column and h the column height, in. The idealized mass equals the sum of the masses on the girder and the girder mass. (Weight of columns and walls is assumed negligible.)

The spring and mass in Fig. 5.108*b* and *d* form a one-degree system. The **degree of a system** is determined by the least number of coordinates needed to define the positions of its components. In Fig. 5.108, only the coordinate y is needed to locate the mass and determine the state of the spring. In a two-degree system, such as one comprising two masses connected to each other and to the ground by springs and capable of movement in only one direction, two coordinates are required to locate the masses.

If the mass with weight W, lb, in Fig. 5.108 is isolated, as shown in Fig. 5.108*e* it will be in dynamic equilibrium under the action of the spring force $-ky$ and the inertia force $(d^2y/dt^2)(W/g)$. Hence, the equation of motion is

$$\frac{W}{g}\frac{d^2y}{dt^2} + ky = 0 \tag{5.235}$$

where y = displacement of mass, in, measured from rest position. Equation (5.235) may be written in the more convenient form

$$\frac{d^2y}{dt^2} + \frac{kg}{W}y = \frac{d^2y}{dt^2} + \omega^2 y = 0 \tag{5.236}$$

The solution is

$$y = A \sin \omega t + B \cos \omega t \tag{5.237}$$

where A and B are constants to be determined from initial conditions of the system, and

$$\omega = \sqrt{\frac{kg}{W}} \tag{5.238}$$

is the natural circular frequency, rad/s.

The motion defined by Eq. (5.237) is harmonic. Its natural period, s, is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{W}{kg}} \tag{5.239}$$

Its natural frequency, Hz, is

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{kg}{W}} \tag{5.240}$$

If, at time $t = 0$, the mass has an initial displacement y_0 and velocity v_0, substitution in Eq. (5.237) yields $A = v_0/\omega$ and $B = y_0$. Hence, at any time t, the mass is completely located by

$$y = \frac{v_0}{\omega}\sin \omega t + y_0 \cos \omega t \tag{5.241}$$

The stress in the spring can be computed from the displacement y.

Vibrations of Lumped Masses. In multiple-degree systems, an independent differential equation of motion can be written for each degree of freedom. Thus, in an N-degree system with N masses, weighing $W_1, W_2, \ldots, W_N$ lb, and N^2 springs with constants k_{rj} ($r = 1, 2, \ldots, N$; $j = 1, 2, \ldots, N$), there are N equations of the form

$$\frac{W_r}{g}\frac{d^2y_r}{dt^2} + \sum_{j=1}^{N} k_{rj}y_j = 0 \qquad r = 1, 2, \ldots, N \tag{5.242}$$

Simultaneous solution of these equations reveals that the motion of each mass can be resolved into N harmonic components. They are called the fundamental, second third, etc., harmonics. Each set of harmonics for all the masses is called a **normal mode** of vibration.

There are as many normal modes in a system as degrees of freedom. Under certain circumstances, the system could vibrate freely in any one of these modes. During any such vibration, the ratio of displacement of any two of the masses remains constant. Hence, the solution of Eqs. (5.242) take the form

$$y_r = \sum_{n=1}^{N} a_{rn} \sin \omega_n(t + \tau_n) \tag{5.243}$$

where a_{rn} and τ_n are constants to be determined from the initial conditions of the system and ω_n is the natural circular frequency for each normal mode.

To determine ω_n, set $y_1 = A_1 \sin \omega t$; $y_2 = A_2 \sin \omega t \ldots$. Then, substitute these values of y_r and their second derivatives in Eqs. (5.242). After dividing each equation by $\sin \omega t$, the following N equations result:

$$\begin{aligned}
\left(k_{11} - \frac{W_1}{g}\omega^2\right)A_1 + k_{12}A_2 + \cdots + k_{1N}A_N &= 0 \\
k_{21}A_1 + \left(k_{22} - \frac{W_2}{g}\right)A_2 + \cdots + k_{2N}A_N &= 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots &\\
k_{N1}A_1 + k_{N2}A_2 + \cdots + \left(k_{NN} - \frac{W_N}{g}\omega^2\right)A_N &= 0
\end{aligned} \tag{5.244}$$

If there are to be nontrivial solutions for the amplitudes $A_1, A_2, \ldots, A_N$, the determinant of their coefficients must be zero. Thus,

$$\begin{bmatrix} k_{11} - \dfrac{W_1}{g}\omega^2 & k_{12} & \cdots & k_{1N} \\ k_{21} & k_{22} - \dfrac{W_2}{g}\omega^2 & \cdots & k_{2N\,N} \\ \cdots & \cdots & \cdots & \cdots \\ k_{N1} & k_{N2} & \cdots & k_{NN} - \dfrac{W}{g}\omega^2 \end{bmatrix} = 0 \quad (5.245)$$

Solution of this equation for ω yields one real root for each normal mode. And the natural period for each normal mode can be obtained from Eq. (5.239).

If ω for a normal mode now is substituted in Eqs. (5.244), the amplitudes A_1, $A_2, \ldots, A_N$ for that mode can be computed in terms of an arbitrary value, usually unity, assigned to one of them. The resulting set of modal amplitudes defines the **characteristic shape** for that mode.

The normal modes are mutually orthogonal; that is,

$$\sum_{r=1}^{N} W_r A_{rn} A_{rm} = 0 \quad (5.246)$$

where W_r is the rth mass out of a total of N, A represents the characteristic amplitude of a normal mode, and n and m identify any two normal modes. Also, for a total of S springs

$$\sum_{s=1}^{S} k_s y_{sn} y_{sm} = 0 \quad (5.247)$$

where k_s is the constant for the sth spring and y represents the spring distortion.

When there are many degrees of freedom, this procedure for analyzing free vibration becomes very lengthy. In such cases, it may be preferable to solve Eqs. (5.244) by numerical, trial-and-error procedures, such as the Stodola-Vianello method. In that method, the solution converges first on the highest or lowest mode. Then, the other modes are determined by the same procedure after elimination of one of the equations by use of Eq. (5.246). The procedure requires assumption of a characteristic shape, a set of amplitudes A_{r1}. These are substituted in one of Eqs. (5.244) to obtain a first approximation of ω^2. With this value and with $A_{N1} = 1$, the remaining $N - 1$ equations are solved to obtain a new set of A_{r1}. Then, the procedure is repeated until assumed and final characteristic amplitudes agree.

Because even this procedure is very lengthy for many degrees of freedom, the Rayleigh approximate method may be used to compute the fundamental mode. The frequency obtained by this method, however, may be a little on the high side.

The Rayleigh method also starts with an assumed set of characteristic amplitudes A_{r1} and depends for its success on the small error in natural frequency produced by a relatively large error in the shape assumption. Next, relative inertia forces acting at each mass are computed: $F_r = W_r A_{r1} / A_{N1}$, where A_{N1} is the assumed displacement at one of the masses. These forces are applied to the system as a static load and displacements B_{r1} due to them calculated. Then, the natural frequency can be obtained from

$$\omega^2 = \frac{g \sum_{r=1}^{N} F_r B_{r1}}{\sum_{r=1}^{N} W_r B_{r1}^2} \tag{5.248}$$

where g is the acceleration due to gravity, 386 in/s^2. For greater accuracy, the computation can be repeated with B_{r1} as the assumed characteristic amplitudes.

When the Rayleigh method is applied to beams, the characteristic shape assumed initially may be chosen conveniently as the deflection curve for static loading.

The Rayleigh method may be extended to determination of higher modes by the Schmidt orthogonalization procedure, which adjusts assumed deflection curves to satisfy Eq. (5.246). The procedure is to assume a shape, remove components associated with lower modes, then use the Rayleigh method for the residual deflection curve. The computation will converge on the next higher mode. The method is shorter than the Stodola-Vianello procedure when only a few modes are needed.

For example, suppose the characteristic amplitudes A_{r1} for the fundamental mode have been obtained and the natural frequency for the second mode is to be computed. Assume a value for the relative deflection of the rth mass A_{r2}. Then, the shape with the fundamental mode removed will be defined by the displacements

$$a_{r2} = A_{r2} - c_1 A_{r1} \tag{5.249}$$

where c_1 is the participation factor for the first mode.

$$c_1 = \frac{\sum_{r=1}^{N} W_r A_{r2} A_{r1}}{\sum_{r=1}^{N} W_r A_{r1}^2} \tag{5.250}$$

Substitute a_{r2} for B_{r1} in Eq. (5.248) to find the second-mode frequency and, from deflections produced by $F_r = W_r a_{r2}$, an improved shape. (For more rapid covergence, A_{r2} should be selected to make c_1 small.) The procedure should be repeated, starting with the new shape.

For the third mode, assume deflections A_{r3} and remove the first two modes:

$$A_{r3} = A_{r3} - c_1 A_{r1} - c_2 A_{r2} \tag{5.251}$$

The participation factors are determined from

$$c_1 = \frac{\sum_{r=1}^{N} W_r A_{r3} A_{r1}}{\sum_{r=1}^{N} W_r A_{r1}^2} \qquad c_2 = \frac{\sum_{r=1}^{N} W_r A_{r3} A_{r2}}{\sum_{r=1}^{N} W_r A_{r2}^2} \tag{5.252}$$

Use a_{r3} to find an improved shape and the third-mode frequency.

Vibrations of Distributed Masses. For some structures with mass distributed throughout, it sometimes is easier to solve the dynamic equations based on distributed mass than the equations based on equivalent lumped masses. A distributed mass has an infinite number of degrees of freedom and normal modes. Every particle in it can be considered a lumped mass on springs connected to other particles. Usually, however, only the fundamental mode is significant, though sometimes the second and third modes must be taken into account.

For example, suppose a beam weighs w lb/lin ft and has a modulus of elasticity E, psi, and moment of inertia I, in^4. Let y be the deflection at a distance x from one end. Then, the equation of motion is

$$EI \frac{\partial^4 y}{\partial x^4} + \frac{w}{g}\frac{\partial^2 y}{\partial t^2} = 0 \tag{5.253}$$

(This equation ignores the effects of shear and rotational inertia.) The deflection y_n for each mode, to satisfy the equation, must be the product of a harmonic function of time $f_n(t)$ and of the characteristic shape $Y_n(x)$, a function of x with undetermined amplitude. The solution is

$$f_n(t) = c_1 \sin \omega_n t + c_2 \cos \omega_n t \tag{5.254}$$

where ω_n is the natural circular frequency and n indicates the mode, and

$$Y_n(x) = A_n \sin \beta_n x + B_n \cos \beta_n x + C_n \sinh \beta_n x + D_n \cosh \beta_n x \tag{5.255}$$

where

$$\beta_n = \sqrt[4]{\frac{w\omega_n^2}{EIg}} \tag{5.256}$$

For a simple beam, the boundary (support) conditions for all values of time t are $y = 0$ and bending moment $M = EI\, \partial^2 y/\partial x^2 = 0$. Hence, at $x = 0$ and $x = L$, the span length, $Y_n(x) = 0$ and $d_2 Y_n/dx^2 = 0$. These conditions require that

$$B_n = C_n = D_n = 0 \qquad \beta_n = \frac{n\pi}{L}$$

to satisfy Eq. (5.255). Hence, according to Eq. (5.256), the natural circular frequency for a simply supported beam is

$$\omega_n = \frac{n^2\pi^2}{L^2}\sqrt{\frac{EIg}{w}} \tag{5.257}$$

The characteristic shape is defined by

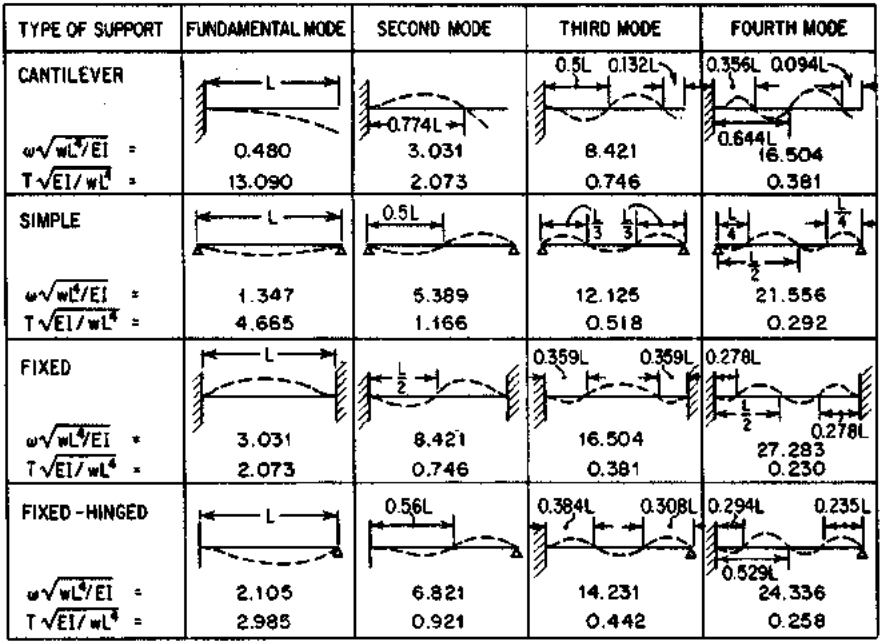
$$Y_n(x) = \sin \frac{n\pi x}{L} \tag{5.258}$$

The constants c_1 and c_2 in Eq. (5.254) are determined by the initial conditions of the disturbance. Thus, the total deflection, by superposition of modes, is

$$y = \sum_{n=1}^{\infty} A_n(t) \sin \frac{n\pi x}{L} \tag{5.259}$$

where $A_n(t)$ is determined by the load (see Art. 5.18.4).

Equations (5.254) to (5.256) apply to spans with any type of end restraints. Figure 5.109 shows the characteristic shape and gives constants for determination



TYPE OF SUPPORT		FUNDAMENTAL MODE	SECOND MODE	THIRD MODE	FOURTH MODE
CANTILEVER	$\omega\sqrt{wL^4/EI}$ =	0.480	3.031	8.421	16.504
	$T\sqrt{EI/wL^4}$ =	13.090	2.073	0.746	0.381
SIMPLE	$\omega\sqrt{wL^4/EI}$ =	1.347	5.389	12.125	21.556
	$T\sqrt{EI/wL^4}$ =	4.665	1.166	0.518	0.292
FIXED	$\omega\sqrt{wL^4/EI}$ =	3.031	8.421	16.504	27.283
	$T\sqrt{EI/wL^4}$ =	2.073	0.746	0.381	0.230
FIXED-HINGED	$\omega\sqrt{wL^4/EI}$ =	2.105	6.821	14.231	24.336
	$T\sqrt{EI/wL^4}$ =	2.985	0.921	0.442	0.258

FIGURE 5.109 Coefficients for computing natural circular frequencies ω and natural periods of vibration T, s, of prismatic beams. w = weight of beam, lb/lin ft; L = span, ft; E = modulus of elasticity of the beam material, psi; I = moment of inertia of the beam cross section, in^4.

of natural circular frequency ω and natural period T for the first four modes of cantilever simply supported, fixed-end, and fixed-hinged beams. To obtain ω, select the appropriate constant from Fig. 5.109 and multiply it by $\sqrt{EI/wL^4}$. where L = span of beam, ft. To get T, divide the appropriate constant by $\sqrt{EI/wL^4}$.

To determine the characteristic shapes and natural periods for beams with variable cross section and mass, use the Rayleigh method. Convert the beam into a lumped-mass system by dividing the span into elements and assuming the mass of each element to be concentrated at its center. Also, compute all quantities, such as deflection and bending moment, at the center of each element. Start with an assumed characteristic shape and apply Eq. (5.255).

Methods are available for dynamic analysis of continuous beams. (R. Clough and J. Penzien, "Dynamics of Structures," McGraw-Hill Book Company, New York; D. G. Fertis and E. C. Zobel, "Transverse Vibration Theory," The Ronald Press Company, New York.) But even for beams with constant cross section, these procedures are very lengthy. Generally, approximate solutions are preferable.

(J. M. Biggs, "Introduction to Structural Dynamics," McGraw-Hill Book Company, New York; N. M. Newmark and E. Rosenblueth, "Fundamentals of Earthquake Engineering," Prentice-Hall, Englewood Cliffs, N.J.)

5.18.3 Impact and Sudden Loads

Under impact, there is an abrupt exchange or absorption of energy and drastic change in velocity. Stresses caused in the colliding members may be several times larger than stresses produced by the same weights applied statically.

An approximation of impact stresses in the elastic range can be made by neglecting the inertia of the body struck and the effect of wave propagation and assuming that the kinetic energy is converted completely into strain energy in that body. Consider a prismatic bar subjected to an axial impact load in tension. The energy absorbed per unit of volume when the bar is stressed to the proportional limit is called the **modulus of resilience.** It is given by $f_y^2/2E$, where f_y is the yield stress and E the modulus of elasticity, both in psi.

Below the proportional limit, the unit stress, psi, due to an axial load U, in-lb, is

$$f = \sqrt{\frac{2UE}{AL}} \tag{5.260}$$

where A is the cross-sectional area, in^2, and L the length of bar, in. This equation indicates that, for a given unit stress, energy absorption of a member may be improved by increasing its length or area. Sharp changes in cross section should be avoided, however, because of associated high stress concentrations. Also, uneven distribution of stress in a member due to changes in section should be avoided. For example, if part of a member is given twice the diameter of another part, the stress under axial load in the larger portion is one-fourth that in the smaller. Since the energy absorbed is proportional to the square of the stress, the energy taken per unit of volume by the larger portion is therefore only one-sixteenth that absorbed by the smaller. So despite the increase in volume caused by doubling of the diameter, the larger portion absorbs much less energy than the smaller. Thus, energy absorption would be larger with a uniform stress distribution throughout the length of the member.

Impact on Short Members. If a static axial load W would produce a tensile stress f' in the bar and an elongation e', in, then the axial stress produced in a short member when W falls a distance h, in, is

$$f = f' + f'\sqrt{1 + \frac{2h}{e'}} \tag{5.261}$$

if f is within the proportional limit. The elongation due to this impact load is

$$e = e' + e'\sqrt{1 + \frac{2h}{e'}} \tag{5.262}$$

These equations indicate that the stress and deformation due to an energy load may be considerably larger than those produced by the same weight applied gradually.

The same equations hold for a beam with constant cross section struck by a weight at midspan, except that f and f' represent stresses at midspan and e and e', midspan deflections.

According to Eqs. (5.261) and (5.262), a sudden load ($h = 0$) causes twice the stress and twice the deflection as the same load applied gradually.

Impact on Long Members. For very long members, the effect of wave propagation should be taken into account. Impact is not transmitted instantly to all parts of the struck body. At first, remote parts remain undisturbed, while particles struck accelerate rapidly to the velocity of the colliding body. The deformations produced move through the struck body in the form of elastic waves. The waves travel with a constant velocity, ft/s,

$$c = 68.1 \sqrt{\frac{E}{\rho}} \tag{5.263}$$

where E = modulus of elasticity, psi
p = density of the struck body, lb/ft^3

If an impact imparts a velocity v, ft/s, to the particles at one end of a prismatic bar, the stress, psi, at that end is

$$f = E\frac{v}{c} = 0.0147v\sqrt{Ep} = 0.000216pcv \tag{5.264}$$

if f is in the elastic range. In a compression wave, the velocity of the particles is in the direction of the wave. In a tension wave, the velocity of the particles is in the direction opposite the wave.

In the plastic range, Eqs. (6.263) and (5.264) hold, but with E as the tangent modulus of elasticity. Hence, c is not a constant and the shape of the stress wave changes as it moves. The elastic portion of the stress wave moves faster than the wave in the plastic range. Where they overlap, the stress and irrecoverable strain are constant.

(The impact theory is based on an assumption difficult to realize in practice—that contact takes place simultaneously over the entire end of the bar.)

At the free end of a bar, a compressive stress wave is reflected as an equal tension wave, and a tension wave as an equal compression wave. The velocity of the particles there equals $2v$.

At a fixed end of a bar, a stress wave is reflected unchanged. The velocity of the particles there is zero, but the stress is doubled, because of the superposition of the two equal stresses on reflection.

For a bar with a fixed end struck at the other end by a moving mass weighing W_m lb, the initial compressive stress, psi, is

$$f_o = 0.0147v_o\sqrt{Ep} \tag{5.265}$$

where v_o is the initial velocity of the particles, ft/s, at the impacted end of the bar and E and p, the modulus of elasticity, psi, and density, lb/ft^3, of the bar. As the velocity of W_m decreases, so does the pressure on the bar. Hence, decreasing compressive stresses follow the wave front. At any time $t < 2L/c$, where L is the length of the bar, in, the stress at the struck end is

$$f = f_o e^{-2\alpha t/\tau} \tag{5.266}$$

where e = 2.71828, α is the ratio of W_b, the weight of the bar, to W_m, and $\tau = 2L/c$.

When $t = \tau$, the wave front with stress f_o arrives back at the struck end, assumed still to be in contact with the mass. Since the velocity of the mass cannot change suddenly, the wave will be reflected as from a fixed end. During the second interval, $\tau < t < 2\tau$, the compressive stress is the sum of two waves moving away from the struck end and one moving toward this end.

Maximum stress from impact occurs at the fixed end. For α greater than 0.2, this stress is

$$f = 2f_o(1 + e^{-2\alpha}) \tag{5.267}$$

For smaller values of α, it is given approximately by

$$f = f_o\left(1 + \sqrt{\frac{1}{\alpha}}\right) \tag{5.268}$$

Duration of impact, time it takes for the impact stress at the struck end to drop to zero, is approximately

$$T = \frac{\pi L}{c\sqrt{\alpha}} \tag{5.269}$$

for small values of α.

When W_m is the weight of a falling body, velocity at impact is $\sqrt{2gh}$, when it falls a distance h, in. Substitution in Eq. (5.265) yields $f_o = \sqrt{2EhW_b/AL}$, since $W_b = pAL$ is the weight of the bar. Putting $W_b = \alpha W_m$; $W_m/A = f'$, the stress produced by W_m when applied gradually, and $E = f'L/e'$, where e' is the elongation for the static load, gives $f_o = f'\sqrt{2h\alpha/e'}$. Then, for values of α smaller than 0.2, the maximum stress, from Eq. (5.268), is

$$f = f'\left(\sqrt{\frac{2h\alpha}{e'}} + \sqrt{\frac{2h}{e'}}\right) \tag{5.270}$$

For larger values of α, the stress wave due to gravity acting on W_m during impact should be added to Eq. (5.267). Thus, for α larger than 0.2,

$$f = 2f'(1 - e^{-2\alpha}) + 2f\sqrt{\frac{2h\alpha}{e'}}(1 + e^{-2\alpha}) \tag{5.271}$$

Equations (5.270) and (5.271) correspond to Eq. (5.261), which was developed without wave effects being taken into account. For a sudden load, $h = 0$, Eq. (5.271) gives for the maximum stress $2f'(1 - e^{-2\alpha})$, not quite double the static stress, the result indicated by Eq. (5.261). (See also Art. 5.18.4.)

(S. Timoshenko and J. N. Goodier, "Theory of Elasticity," McGraw-Hill Book Company, New York; S. Timoshenko and D. H. Young, "Engineering Mechanics," John Wiley & Sons, Inc., New York.)

5.18.4 Dynamic Analysis of Simple Structures

Articles 5.181 to 5.18.3 present a theoretic basis for analysis of structures under dynamic loads. As noted in Art. 5.18.2, an approximate solution based on an idealized representation of an actual member of structure is advisable for dynamic analysis and design. Generally, the actual structure may be conveniently represented by a system of masses and massless springs, with additional resistances to account for damping. In simple cases, the masses may be set equal to the actual masses; otherwise, equivalent masses may be substituted for the actual masses (Art. 5.18.6). The spring constants are the ratios of forces to deflections (see Art. 5.18.2).

Usually, for structural purposes the data sought are the maximum stresses in the springs and their maximum displacements and the time of occurrence of the maximums. This time is generally computed in terms of the natural period of vibration of the member or structure, or in terms of the duration of the load. Maximum displacement may be calculated in terms of the deflection that would result if the load were applied gradually.

The term D by which the static deflection e', spring forces, and stresses are multiplied to obtain the dynamic effects is called the **dynamic load factor.** Thus, the dynamic displacement is

$$y = De' \tag{5.272}$$

And the maximum displacement y_m is determined by the maximum dynamic load factor D_m, which occurs at time t_m.

One-Degree Systems. Consider the one-degree-of-freedom system in Fig. 5.110*a*. It may represent a weightless beam with a mass weighing W lb applied at midspan and subjected to a varying force $F_o f(t)$, or a rigid frame with a mass weighing W lb at girder level and subjected to this force. The force is represented by an arbitrarily chosen constant force F_o times $F(t)$, a function of time.

If the system is not damped, the equation of motion in the elastic range is

$$\frac{W}{g}\frac{d^2y}{dt^2} + ky = F_o f(t) \tag{5.273}$$

where k is the spring constant and g the acceleration due to gravity, 386 in/s^2. The solution consists of two parts. The first, called the complementary solution, is obtained by setting $f(t) = 0$. This solution is given by Eq. (5.237). To it must be added the second part, the particular solution, which satisfies Eq. (5.273).

The general solution of Eq. (5.273), arrived at by treating an element of the force-time curve (Fig. 5.111*b*) as an impulse, is

$$y = y_o \cos \omega t + \frac{v_o}{\omega} \sin \omega t + e'\omega \int_0^t f(\tau) \sin \omega(t - \tau)\, d\tau \tag{5.274}$$

where y = displacement of mass from equilibrium position, in
y_o = initial displacement of mass ($t = 0$), in
$\omega = \sqrt{kg/W}$ = natural circular frequency of free vibration

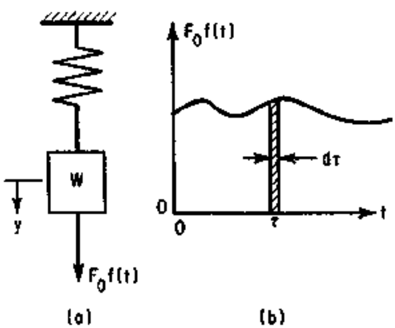


FIGURE 5.110 One-degree system acted on by a force varying with time.

k = spring constant = force producing unit deflection, lb/in
v_o = initial velocity of mass, in/s
e' = F_o/k = displacement under static load, in

A closed solution is possible if the integral can be evaluated.

Assume, for example, the mass is subjected to a suddenly applied force F_o that remains constant (Fig. 5.111*a*). If y_o and v_o are initially zero, the displacement y of the mass at any time t can be obtained from the integral in Eq. (5.274) by setting $f(\tau) = 1$:

$$y = e'\omega \int_0^t \sin \omega(t - \tau)\, d\tau = e'(1 - \cos \omega t) \tag{5.275}$$

This equation indicates that the dynamic load factor $D = 1 - \cos \omega t$. It has a maximum value $D_m = 2$ when $t = \pi/\omega$. Figure 5.111*b* shows the variation of displacement with time.

Multidegree Systems. A multidegree lumped-mass system may be analyzed by the modal method after the natural frequencies of the normal modes have been determined (Art. 5.18.2). This method is restricted to linearly elastic systems in which the forces applied to the masses have the same variation with time. For other cases, numerical analysis must be used.

In the modal method, each normal mode is treated as an independent one-degree system. For each degree of the system, there is one normal mode. A natural frequency and a characteristic shape are associated with each mode. In each mode, the ratio of the displacements of any two masses is constant with time. These ratios define the characteristic shape. The modal equation of motion for each mode is

$$\frac{d^2A_n}{dt^2} + \omega_n^2 A_n = \frac{gf(t) \sum_{r=1}^{j} F_r \phi_{rn}}{\sum_{r=1}^{j} W_r \phi_{rn}^2} \tag{5.276}$$

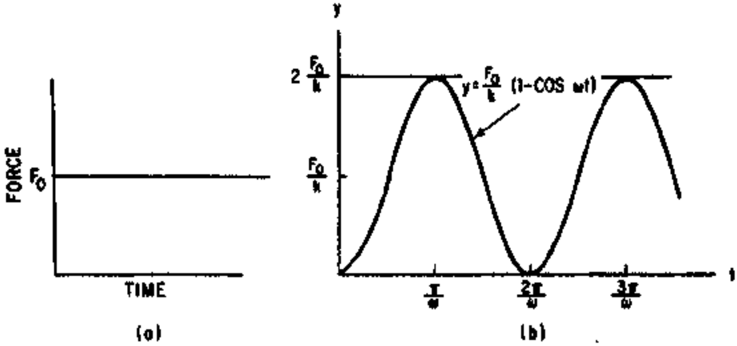


FIGURE 5.111 Harmonic motion. (*a*) Constant force applied to an undamped one-degree system, such as the one in Fig. 5.110*a*. (*b*) Displacements vary with time like a cosine curve.

where A_n = displacement in the nth mode of an arbitrarily selected mass
ω_n = natural frequency of the nth mode
$F_r f(t)$ = varying force applied to the rth mass
W_r = weight of the rth mass
j = number of masses in the system
ϕ_{rn} = ratio of the displacement in the nth mode of the rth mass to A_n
g = acceleration due to gravity

We define the modal static deflection as

$$A'_n = \frac{g \sum_{r=1}^{j} F_r \phi_{rn}}{\omega_n^2 \sum_{r=1}^{j} W_r \phi_{rn}^2} \tag{5.277}$$

Then, the response for each mode is given by

$$A_n = D_n A'_n \tag{5.278}$$

where D_n = dynamic load factor.

Since D_n depends only on ω_n and the variation of force with time $f(t)$, solutions for D_n obtained for one-degree systems also apply to multidegree systems. The total deflection at any point is the sum of the displacements for each mode, $\Sigma A_n \phi_{rn}$, at that point.

Beams. The response of beams to dynamic forces can be determined in a similar way. The modal static deflection is defined by

$$A'_n = \frac{\int_0^L p(x)\phi_n(x)\,dx}{\omega_n^2 \dfrac{w}{g} \int_0^L \phi_n^2(x)\,dx} \tag{5.279}$$

where $p(x)$ = load distribution on the span [$p(x)f(x)$ is the varying force]
$\phi_n(x)$ = characteristic shape of the nth mode (see Art. 5.18.2)
L = span length
w = uniformly distributed weight on the span

The response of the beam then is given by Eq. (5.278), and the dynamic deflection is the sum of the modal components, $\Sigma A_n \phi_n(x)$.

Nonlinear Responses. When the structure does not react linearly to loads, the equations of motion can be solved by numerical analysis if resistance is a unique function of displacement. Sometimes, the behavior of the structure can be represented by an idealized resistance-displacement diagram that makes possible a solution in closed form. Figure 5.112*a* shows such a diagram.

Elastic-Plastic Responses. Resistance is assumed linear ($R = ky$) in Fig. 5.112*a* until a maximum R_m is reached. After that, R remains equal to R_m for increases in y substantially larger than the displacement y_e at the elastic limit. Thus, some portions of the structure deform into the plastic range. Figure 5.112*a*, therefore, may be used for ductile structures only rarely subjected to severe dynamic loads. When

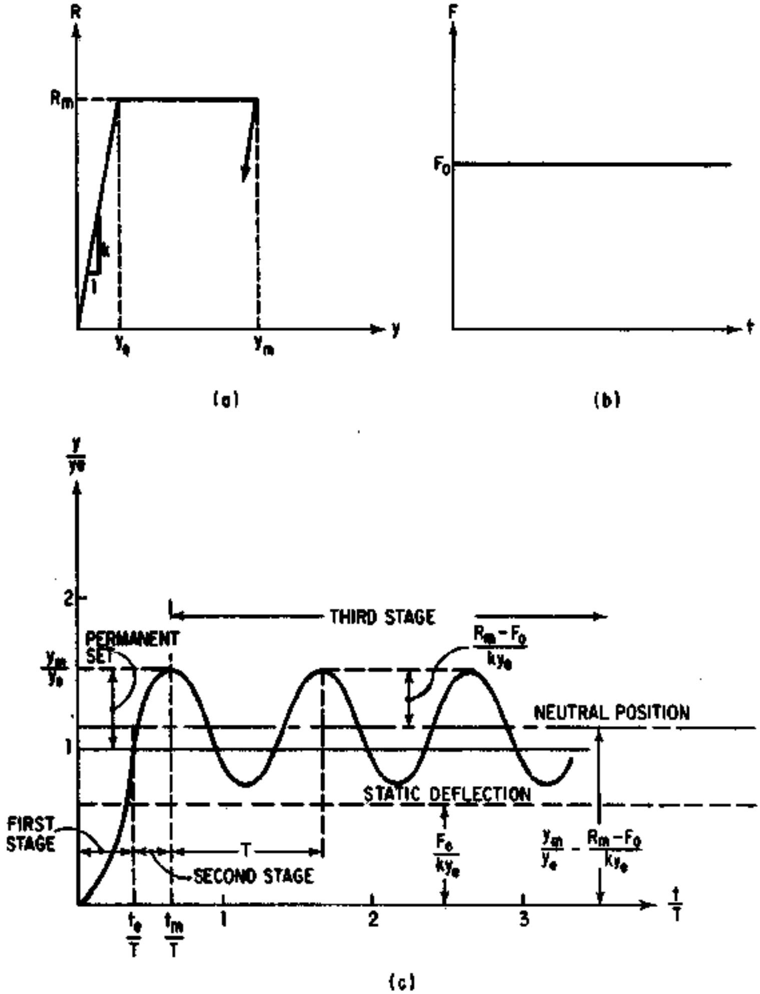


FIGURE 5.112 Response in the plastic range of a one-degree system with resistance characteristics indicated in (*a*) and subjected to a constant force (*b*) is shown in (*c*).

this diagram can be used for designing such structures, more economical designs can be produced than for structures limited to the elastic range, because of the high energy-absorption capacity of structures in the plastic range.

For a one-degree system, Eq. (5.273) can be used as the equation of motion for the initial sloping part of the diagram (elastic range). For the second stage, $y_e < y < y_m$, where y_m is the maximum displacement, the equation is

$$\frac{W}{g}\frac{d^2y}{dt^2} + R_m = F_o f(t) \tag{5.280}$$

For the unloading stage, $y < y_m$, the equation is

$$\frac{W}{g}\frac{d^2y}{dt^2} + R_m - k(y_m - y) = F_o f(t) \tag{5.281}$$

Suppose, for example, the one-degree undamped system in Fig. 5.109*a* behaves in accordance with the bilinear resistance function of Fig. 5.112*a* and is subjected to a suddenly applied constant load (Fig. 5.112*b*). With zero initial displacement and velocity, the response in the first stage ($y < y_e$), according to Eq. (5.281), is

$$y = e'(1 - \cos \omega t_1) \tag{5.282}$$

$$\frac{dy}{dt} = e'\omega \sin \omega t_1 \tag{5.283}$$

Equation (5.275) also indicates that displacement y_e will be reached at a time t_e such that $\cos \omega t_e = 1 - y_e/e'$.

For convenience, let $t_2 = t - t_e$ be the time in the second stage; thus, $t_2 = 0$ at the start of that stage. Since the condition of the system at that time is the same as at the end of the first stage, the initial displacement is y_e and the initial velocity $e'\omega \sin \omega t_e$.

The equation of motion of the second stage is

$$\frac{W}{g}\frac{d^2y}{dt^2} + R_m = F_o \tag{5.284}$$

The solution, taking into account initial conditions for $y_e < y < y_m$ is

$$y = \frac{g}{2W}(F_o - R_m)t_2^2 + e'\omega t_2 \sin \omega t_e + y_e \tag{5.285}$$

Maximum displacement occurs at the time

$$t_m = \frac{W\omega e'}{g(R_m - R_o)} \sin \omega t_e \tag{5.286}$$

and can be obtained by substituting t_m in Eq. (5.285).

The third stage, unloading after y_m has been reached, can be determined from Eq. (5.281) and conditions at the end of the second stage. The response, however, is more easily found by noting that the third stage consists of an elastic, harmonic residual vibration. In this stage the amplitude of vibration is $(R_m - F_o)/k$, since this is the distance between the neutral position and maximum displacement, and in the neutral position the spring force equals F_o. Hence, the response can be obtained directly from Eq. (5.275) by substituting $y_m - (R_m - F_o)/k$ for e', because the neutral position, located at $y = y_m - (R_m - F_o)/k$, occurs when $\omega t_3 = \pi/2$, where $t_3 = t - t_e - t_m$. The solution is

$$y = y_m - \frac{R_m - F_o}{k} + \frac{R_m - F_o}{k} \cos \omega t_3 \tag{5.287}$$

Response in the three stages is shown in Fig. 5.112*c*. In that diagram, however, to represent a typical case, the coordinates have been made nondimensional by expressing y in terms of y_e and the time in terms of T, the natural period of vibration.

(J. M. Biggs, "Introduction to Structural Dynamics," and R. Clough and J. Penzien, "Dynamics of Structures," McGraw-Hill Book Company, New York; D. G. Fertis and E. C. Zobel, "Transverse Vibration Theory," The Ronald Press Company, New York; N. M. Newmark and E. Rosenblueth, "Fundamentals of Earthquake Engineering," Prentice-Hall, Englewood Cliffs, N.J.)

5.18.5 Resonance and Damping

Damping in structures, resulting from friction and other causes, resists motion imposed by dynamic loads. Generally, the effect is to decrease the amplitude and lengthen the period of vibrations. If damping is large enough, vibration may be eliminated.

When maximum stress and displacement are the prime concern, damping may not be of great significance for short-time loads. These maximums usually occur under such loads at the first peak of response, and damping, unless unusually large, has little effect in a short period of time. But under conditions close to resonance, damping has considerable effect.

Resonance is the condition of a vibrating system under a varying load such that the amplitude of successive vibrations increases. Unless limited by damping or changes in the condition of the system, amplitudes may become very large.

Two forms of damping generally are assumed in structural analysis, viscous or constant (Coulomb). For viscous damping, the damping force is taken proportional to the velocity but opposite in direction. For Coulomb damping, the damping force is assumed constant and opposed in direction to the velocity.

Viscous Damping. For a one-degree system (Arts. 5.18.2 to 5.18.4), the equation of motion for a mass weighing W lb and subjected to a force F varying with time but opposed by viscous damping is

$$\frac{W}{g}\frac{d^2y}{dt^2} + ky = F - c\frac{dy}{dt} \tag{5.288}$$

where y = displacement of the mass from equilibrium position, in
k = spring constant, lb/in
t = time, s
c = coefficient of viscous damping
g = acceleration due to gravity = 386 in/s^2

Let us set $\beta = cg/2W$ and consider those cases in which $\beta\ \omega$, the natural circular frequency [Eq. (5.238)], to eliminate unusually high damping (overdamping). Then, for initial displacement y_o and velocity v_o, the solution of Eq. (5.288) with $F = 0$ is

$$y = e^{-\beta t}\left(\frac{v_o + \beta y_o}{\omega_d}\sin\omega_d t + y_o\cos\omega_d t\right) \tag{5.289}$$

where $\omega_d = \sqrt{\omega^2 - \beta^2}$ and $e = 2.71828$. Equation (5.289) represents a decaying harmonic motion with β controlling the rate of decay and ω_d the natural frequency of the damped system.

When $\beta = \omega$

$$y = e^{-\omega t}[v_o t + (1 + \omega t)y_o] \tag{5.290}$$

which indicates that the motion is not vibratory. Damping producing this condition is called critical, and, from the definition of β, the critical coefficient is

$$c_d = \frac{2W\beta}{g} = \frac{2W\omega}{g} = 2\sqrt{\frac{kW}{g}} \tag{5.291}$$

Damping sometimes is expressed as a percent of critical (β as a percent of ω).

For small amounts of viscous damping, the damped natural frequency is approximately equal to the undamped natural frequency minus $\frac{1}{2}\beta^2/\omega$. For example, for 10% critical damping ($\beta = 0.1\omega$), $\omega_d = \omega[1 - \frac{1}{2}(0.1)^2] = 0.995\omega$. Hence, the decrease in natural frequency due to small amounts of damping generally can be ignored.

Damping sometimes is measured by **logarithmic decrement,** the logarithm of the ratio of two consecutive peak amplitudes during free vibration.

$$\text{Logarithmic decrement} = \frac{2\pi\beta}{\omega} \tag{5.292}$$

For example, for 10% critical damping, the logarithmic decrement equals 0.2π. Hence, the ratio of a peak to the following peak amplitude is $e^{0.2\pi} = 1.87$.

The complete solution of Eq. (5.288) with initial displacement y_o and velocity v_o is

$$y = e^{-\beta t}\left(\frac{v_o + \beta y_o}{\omega_d}\sin\omega_d t + y_o\cos\omega_d t\right) + e'\frac{\omega^2}{\omega_d}\int_0^t f(\tau)e^{-\beta(t-\tau)}\sin\omega_d(t-\tau)\,d\tau \tag{5.293}$$

where e' is the deflection that the applied force would produce under static loading. Equation (5.293) is identical to Eq. (5.274) when $\beta = 0$.

Unbalanced rotating parts of machines produce pulsating forces that may be represented by functions of the form $F_o \sin \alpha t$. If such a force is applied to an undamped one-degree system. Eq. (5.274) indicates that if the system starts at rest the response will be

$$y = \frac{F_o g}{W}\left(\frac{1/\omega^2}{1 - \alpha^2/\omega^2}\right)\left(\sin\alpha t - \frac{\alpha}{\omega}\sin\omega t\right) \tag{5.294}$$

And since the static deflection would be $F_o/k = F_o g/W\omega^2$, the dynamic load factor is

$$D = \frac{1}{1 - \alpha^2/\omega^2}\left(\sin\alpha t - \frac{\alpha}{\omega}\sin\omega t\right) \tag{5.295}$$

If α is small relative to ω, maximum D is nearly unity; thus, the system is practically statically loaded. If α is very large compared with ω, D is very small; thus, the mass cannot follow the rapid fluctuations in load and remains practically stationary. Therefore, when α differs appreciably from ω, the effects of unbalanced rotating parts are not too serious. But if $\alpha = \omega$, resonance occurs; D increases with time. Hence, to prevent structural damage, measures must be taken to correct the unbalanced parts to change α, or to change the natural frequency of the vibrating mass, or damping must be provided.

The response as given by Eq. (5.294) consists of two parts, the free vibration and the forced part. When damping is present, the free vibration is of the form of

Eq. (5.289) and is rapidly damped out. Hence, the free part is called the **transient response,** and the forced part, the **steady-state response.** The maximum value of the dynamic load factor for the steady-state response D_m is called the **dynamic magnification factor.** It is given by

$$D_m = \frac{1}{\sqrt{(1 - \alpha^2/\omega^2)^2 + (2\beta\alpha/\omega^2)^2}} \tag{5.296}$$

With damping, then, the peak values of D_m occur when $\alpha = \omega\sqrt{1 - \beta^2/\omega^2}$ and are approximately equal to $\omega/2\beta$. For example, for 10% critical damping.

$$D_m = \frac{\omega}{0.2\omega} = 5$$

So even small amounts of damping significantly limit the response at resonance.

Coulomb Damping. For a one-degree system with Coulomb damping, the equation of motion for free vibration is

$$\frac{W}{g}\frac{d^2y}{dt^2} + ky = \pm F_f \tag{5.297}$$

where F_f is the constant friction force and the positive sign applies when the velocity is negative. If initial displacement is y_o and initial velocity is zero, the response in the first half cycle, with negative velocity, is

$$y = \left(y_o - \frac{F_f}{k}\right)\cos\omega t + \frac{F_f}{k} \tag{5.298}$$

equivalent to a system with a suddenly applied constant force. For the second half cycle, with positive velocity, the response is

$$y = \left(-y_o + 3\frac{F_f}{k}\right)\cos\omega\left(t - \frac{\pi}{\omega}\right) - \frac{F_f}{k} \tag{5.299}$$

If the solution is continued with the sign of F_f changing in each half cycle, the results will indicate that the amplitude of positive peaks is given by $y_o - 4nF_f/k$, where n is the number of complete cycles, and the response will be completely damped out when $t = ky_oT/4F_f$, where T is the natural period of vibration, or $2\pi/\omega$.

Analysis of the steady-state response with Coulomb damping is complicated by the possibility of frequent cessation of motion.

(S. Timoshenko, D. H. Young, and W. Weaver, "Vibration Problems in Engineering," 4th ed., John Wiley & Sons, Inc., New York; D. D. Barkan, "Dynamics of Bases and Foundations," McGraw-Hill Book Company; W. C. Hurty and M. F. Rubinstein, "Dynamics of Structures," Prentice-Hall, Englewood Cliffs, N.J.)

5.18.6 Approximate Design for Dynamic Loading

Complex analysis and design methods seldom are justified for structures subject to dynamic loading because of lack of sufficient information on loading, damping, resistance to deformation, and other factors. In general, it is advisable to represent the actual structure and loading by idealized systems that permit a solution in closed form (see Arts. 5.18.1 to 5.18.5).

Whenever possible, represent the actual structure by a one-degree system consisting of an equivalent mass with massless spring. For structures with distributed mass. simplify the analysis in the elastic range by computing the response only for one or a few of the normal modes. In the plastic range, treat each stage—elastic, and plastic—as completely independent; for example, a fixed-end beam may be treated, when in the elastic-plastic stage, as a simply supported beam.

Choose the parameters of the equivalent system to make the deflection at a critical point, such as the location of the concentrated mass, the same as it would be in the actual structure. Stresses in the actual structure should be computed from the deflections in the equivalent system.

Compute an assumed shape factor ϕ for the system from the shape taken by the actual structure under static application of the loads. For example, for a simple beam in the elastic range with concentrated load at midspan, ϕ may be chosen, for $x < L/2$, as $(Cx/L^3)(3L^2 - 4x^2)$, the shape under static loading, and C may be set equal to 1 to make ϕ equal to 1 when $x = L/2$. For plastic conditions (hinge at midspan), ϕ may be taken as Cx/L, and C set equal to 2, to make $\phi = 1$ when $x = L/2$.

For a structure with concentrated forces, let W_r be the weight of the rth mass, ϕ_r the value of ϕ for a specific mode at the location of that mass, and F_r the dynamic force acting on W_r. Then, the equivalent weight of the idealized system is

$$W_e = \sum_{r=1}^{j} W_r \phi_r^2 \tag{5.300}$$

where j is the number of masses. The equivalent force is

$$F_e = \sum_{r=1}^{j} F_r \phi_r \tag{5.301}$$

For a structure with continuous mass, the equivalent weight is

$$W_e = \int w\phi^2 \, dx \tag{5.302}$$

where w is the weight in lb/lin ft. The equivalent force is

$$F_e = \int q\phi \, dx \tag{5.303}$$

for a distributed load q, lb/lin ft.

The resistance of a member or structure is the internal force tending to restore it to its unloaded static position. For most structures, a bilinear resistance function, with slope k up to the elastic limit and zero slope in the plastic range (Fig. 5.112a), may be assumed. For a given distribution of dynamic load, maximum resistance of the idealized system may be taken as the total load with that distribution that the structure can support statically. Similarly, stiffness is numerically equal to the total load with the given distribution that would cause a unit deflection at the point where the deflections in the actual structure and idealized system are equal. Hence, the equivalent resistance and stiffness are in the same ratio to the actual as the equivalent forces to the actual forces.

Let k be the actual spring constant, g acceleration due to gravity, 386 in/s^2, and

$$W' = \frac{W_e}{F_e} \Sigma F \tag{5.304}$$

where ΣF represents the actual total load. Then, the equation of motion of an equivalent one-degree system is

$$\frac{d^2y}{dt^2} + \omega^2 y = g \frac{\Sigma F}{W'} \tag{5.305}$$

and the natural circular frequency is

$$\omega = \sqrt{\frac{kg}{W'}} \tag{5.306}$$

The natural period of vibration equals $2\pi/\omega$. Equations (5.305) and (5.306) have the same form as Eqs. (5.236), (5.238), and (5.273). Consequently, the response can be computed as indicated in Arts. 5.18.2 to 5.18.4.

Whenever possible, select a load-time function for ΣF to permit use of a known solution.

For preliminary design of a one-degree system loaded into the plastic range by a suddenly applied force that remains substantially constant up to the time of maximum response, the following approximation may be used for that response:

$$y_m = \frac{y_e}{2(1 - F_o/R_m)} \tag{5.307}$$

where y_e is the displacement at the elastic limit, F_o the average value of the force, and R_m the maximum resistance of the system. This equation indicates that for purely elastic response, R_m must be twice F_o; whereas, if y_m is permitted to be large, R_m may be made nearly equal to F_o, with greater economy of material.

For preliminary design of a one-degree system subjected to a sudden load with duration t_d less than 20% of the natural period of the system, the following approximation can be used for the maximum response:

$$y_m = \frac{1}{2} y_e \left[\left(\frac{F_o}{R_m} \omega t_d \right)^2 + 1 \right] \tag{5.308}$$

where F_o is the maximum value of the load and ω the natural frequency. This equation also indicates that the larger y_m is permitted to be, the smaller R_m need be.

For a beam, the spring force of the equivalent system is not the actual force, or reaction, at the supports. The real reactions should be determined from the dynamic equilibrium of the complete beam. This calculation should include the inertia force, with distribution identical with the assumed deflected shape of the beam. For example, for a simply supported beam with uniform load, the dynamic reaction in the elastic range is $0.39R + 0.11F$, where R is the resistance, which varies with time, and $F = qL$ is the load. For a concentrated load F at midspan, the dynamic reaction is $0.78R - 0.28F$. And for concentrated loads $F/2$ at each third point, it is $0.62R - 0.12F$. (Note that the sum of the coefficients equals 0.50, since the dynamic-reaction equations must hold for static loading, when $R = F$.) These expressions also can be used for fixed-end beams without significant error. If high accuracy is not required, they also can be used for the plastic range.

5.19 EARTHQUAKE LOADS

The seismic loads on the structure during an earthquake result from inertia forces which were created by ground accelerations. The magnitude of these loads is a function of the following factors: mass of the building, the dynamic properties of the building, the intensity, duration, and frequency content of the ground motion, and soil-structure interaction. In recent years, a lot of achievements have been made to incorporate these influential factors into building codes accurately as well as practically. The basis for IBC 2000 seismic provisions is the 1997 NEHRP "Recommended Provisions for the Development of Seismic Regulations for New Buildings and Other Structures" (FEMA 302). The National Earthquake Hazard Reduction Program (NEHRP) is managed by the Federal Emergency Management Agency (FEMA).

In IBC 2000, the seismic loads are on a strength level limit state rather than on a service load level, which was used in UBC 94 and prior versions. The seismic limit state is based upon system performance, not member performance, and considerable energy dissipation through repeated cycles of inelastic straining is assumed.

5.19.1 Criteria Selection

In IBC 2000, the following basic information is required to determine the seismic loads:

1. *Seismic Use Group* According to the nature of Building Occupancy, each structure is assigned a Seismic Use Group (I, II, or III) and a corresponding Occupancy Importance (I) factor ($I = 1.0$, 1.25, or 1.5).

   Seismic Use Group I structures are those not assigned to either Seismic Use Group II or III. Seismic Use Group II are structures whose failure would result in a substantial public hazard due to occupancy or use. Seismic Use Group III is assigned to structures for which failure would result in loss of essential facilities required for post-earthquake recovery and those containing substantial quantities of hazardous substances.
2. *Site Class* Based on the soil properties, the site of building is classified as A, B, C, D, E, or F to reflect the soil-structure interaction. Refer to IBC 2000 for Site Class definition.
3. *Spectral Response Accelerations* S_S *and* S_1 The spectral response seismic design maps reflect seismic hazards on the basis of contours. They provide the maximum considered earthquake spectral response acceleration at short period S_S and at 1-second period S_1. They are for Site Class B, with 5% of critical damping. Refer to the maps in IBC 2000.
4. *Basic Seismic-Force-Resisting System* Different types of structural system have different energy-absorbing characteristics. The response modification coefficient

R in Table 5.9 is used to account for these characteristics. Systems with higher ductility have higher R values.

With the above basic parameters available, the following design and analysis criteria can be determined.

Seismic Design Category. The Seismic Design Category is based on the seismic group and the design spectral response acceleration coefficients, S_{DS} and S_{D1}, which will be explained later. The Seismic Design Category for a structure can be determined in accordance with Tables 5.10 and 5.11.

Seismic Design Categories are used to determine the permissible structural systems, the limitations on height and irregularity of the structural components that must be designed for seismic resistance and the types of lateral force analysis that must be performed.

Seismic Use Groups I and II structures located on sites with mapped maximum considered earthquake spectral response acceleration at 1-second period S_1, equal to or greater than $0.75g$, shall be assigned to Seismic Design Category E. Seismic Use Group III structures located on such sites shall be assigned to Seismic Design Category F. A structure assigned to Seismic Design Category E or F shall not be sited where there is the potential for an active fault to cause rupture of the ground surface at the structure.

Building Irregularity. Building with irregular shapes, changes in mass from floor to floor, variable stiffness with height, and unusual setbacks do not perform well during earthquakes. Thus, for each type of these irregularities, additional design requirements shall be followed to maintain seismic-resisting capacity. IBC 2000 requires that all buildings be classified as regular or irregular based on the plan and vertical configuration. See Tables 5.12 and 5.13 for classification and corresponding requirements.

Design Requirements for Seismic Design Category A. Structures assigned to Seismic Design Category A need only comply with the following:

- Structure shall be provided with a complete lateral-force-resisting system designed to resist the minimum lateral force, of 1% floor gravity load.

  The gravity load should include the total dead load and other loads listed below.

  - In areas used for storage, a minimum of 25% of the reduced floor live load (floor live load in public garages and open parking structures need not be included)
  - Where an allowance for partition load is included in the floor load design, the actual partition weight or a minimum weight of 10 psf of floor area (whichever is greater)
  - Total operating weight of permanent equipment
  - 20% of flat roof snow load where flat roof snow load exceeds 30 psf
- The direction of application of seismic forces used in design shall be that which will produce the most critical load effect in each component. The design seismic forces are permitted to be applied separately in each of two orthogonal directions and orthogonal effects are permitted to be neglected.
- The effect of this lateral force shall be taken as E in the load combinations. Special seismic load combinations that include E_m need not to be considered.

TABLE 5.9 Design Coefficients and Factors for Basic Seismic-Force-Resisting Systems

Basic seismic-force-resisting system	Response modification coefficient, R	System over-strength factor, Ω_o	Deflection amplification factor, C_d	System limitations and building height limitations (ft) by seismic design category				
				A and B	C	D	E	F
Bearing wall systems								
Ordinary steel braced frames	4	2	3½	NL	NL	160	160	160
Special reinforced concrete shear walls	5½	2½	5	NL	NL	160	160	100
Ordinary reinforced concrete shear walls	4½	2½	4	NL	NL	NP	NP	NP
Detailed plain concrete shear walls	2½	2½	4	NL	NL	NP	NP	NP
Ordinary plain concrete shear walls	1½	2½	1½	NL	NP	NP	NP	NP
Special reinforced masonry shear walls	4	2½	3½	NL	NL	160	160	100
Intermediate reinforced masonry shear walls	3½	2½	3	NL	NL	NP	NP	NP
Ordinary reinforced masonry shear walls	2	2½	1¾	NL	160	NP	NP	NP
Detailed plain masonry shear walls	2	2½	1¾	NL	160	NP	NP	NP
Ordinary plain masonry shear walls	1½	2½	1¼	NL	NP	NP	NP	NP
Light frame walls with shear panels, Wood Structural Panels	6½	3	4	NL	NL	160	160	100
Light frame walls with shear panels—Gypsum Board	2	2½	2	NL	NL	35	NP	NP
Building frame systems								
Steel eccentrically braced frames, nonmoment resisting, connections at columns away from links	7	2	4	NL	NL	160	160	100
Special steel concentrically braced frames	6	2½	5	NL	NL	160	160	100
Ordinary steel concentrically braced frames	5	2	4½	NL	NL	160	100	100
Special reinforced concrete shear walls	6	2½	5	NL	NL	160	160	100

TABLE 5.9 Design Coefficients and Factors for Basic Seismic-Force-Resisting Systems (*Continued*)

Basic seismic-force-resisting system	Response modification coefficient, R	System over-strength factor, Ω_o	Deflection amplification factor, C_d	System limitations and building height limitations (ft) by seismic design category				
				A and B	C	D	E	F
Bearing wall systems								
Ordinary reinforced concrete shear walls	5	2½	4½	NL	NL	NP	NP	NP
Detailed plain concrete shear walls	3	2½	2½	NL	NL	NP	NP	NP
Ordinary plain concrete shear walls	2	2½	2	NL	NP	NP	NP	NP
Composite eccentrically braced frames	8	2	4	NL	NL	160	160	100
Composite concentrically braced frames	5	2	4½	NL	NL	160	160	100
Ordinary composite braced frames	3	2	3	NL	NL	NP	NP	NP
Composite steel plate shear walls	6½	2½	5½	NL	NL	160	160	100
Special composite reinforced concrete shear walls with steel elements	6	2½	5	NL	NL	160	160	100
Ordinary composite reinforced concrete shear walls with steel elements	5	2½	4½	NL	NL	NP	NP	NP
Special reinforced masonry shear walls	5	2½	4	NL	NL	160	160	100
Intermediate reinforced masonry shear walls	4½	2½	4	NL	NL	160	160	100
Ordinary reinforced masonry shear walls	2½	2½	2¼	NL	160	NP	NP	NP
Detailed plain masonry shear walls	2½	2½	2¼	NL	160	NP	NP	NP
Ordinary plain masonry shear walls	1½	2½	1¼	NL	NP	NP	NP	NP
Light frame walls with shear panels	7	2½	4½	NL	NL	160	160	160

TABLE 5.9 Design Coefficients and Factors for Basic Seismic-Force-Resisting Systems (*Continued*)

Basic seismic-force-resisting system	Response modification coefficient, R	System over-strength factor, Ω_o	Deflection amplification factor, C_d	System limitations and building height limitations (ft) by seismic design category				
				A and B	C	D	E	F
Moment resisting frame systems								
Special steel moment frames	8	3	5½	NL	NL	NL	NL	NL
Special steel truss moment frames	7	3	5½	NL	NL	160	100	NP
Intermediate steel moment frames	6	3	5	NL	NL	160	100	NP
Ordinary steel moment frames	4	3	3½	NL	NL	35	NP	NP
Special reinforced concrete moment frames	8	3	5½	NL	NL	NL	NL	NL
Intermediate reinforced concrete moment frames	5	3	4½	NL	NL	NP	NP	NP
Ordinary reinforced concrete moment frames	3	3	2½	NL	NP	NP	NP	NP
Special composite moment frames	8	3	5½	NL	NL	NL	NL	NL
Intermediate composite moment frames	5	3	4½	NL	NL	NP	NP	NP
Composite partially restrained moment frames	6	3	5½	160	160	100	NP	NP
Ordinary composite moment frames	3	3	4	NL	NP	NP	NP	NP
Masonry wall frames	5½	3	5	NL	NL	160	160	100
Dual systems with special moment frames								
Steel eccentrically braced frames, moment-resisting connections, at columns away from links	8	2½	4	NL	NL	NL	NL	NL
Steel eccentrically braced frames, nonmoment-resisting connections, at columns away from links	7	2½	4	NL	NL	NL	NL	NL
Special steel concentrically braced frames	8	2½	6½	NL	NL	NL	NL	NL
Ordinary steel concentrically braced frames	6	2½	5	NL	NL	NL	NL	NL
Special reinforced concrete shear walls	8	2½	6½	NL	NL	NL	NL	NL

TABLE 5.9 Design Coefficients and Factors for Basic Seismic-Force-Resisting Systems (*Continued*)

Basic seismic-force-resisting system	Response modification coefficient, R	System over-strength factor, Ω_o	Deflection amplification factor, C_d	System limitations and building height limitations (ft) by seismic design category				
				A and B	C	D	E	F
Ordinary reinforced concrete shear walls	7	2½	6	NL	NL	NP	NP	NP
Composite eccentrically braced frames	8	2½	4	NL	NL	NL	NL	NL
Composite concentrically braced frames	6	2½	5	NL	NL	NL	NL	NL
Composite steel plate shear walls	8	3	6½	NL	NL	NL	NL	NL
Special composite reinforced concrete shear walls with steel elements	8	3	6½	NL	NL	NL	NL	NL
Ordinary composite reinforced concrete shjear walls with steel elements	7	3	6½	NL	NL	NP	NP	NP
Special reinforced masonry shear walls	7	3	6½	NL	NL	NL	NL	NL
Intermediate reinforced masonry shear walls	6½	3	5½	NL	NL	NL	NP	NP
Dual systems with intermediate moment frames								
Special steel concentrically braced frames	6	2½	5	NL	NL	160	100	NP
Ordinary steel concentrically braced frames	5	2½	4½	NL	NL	160	100	NP
Special reinforced concrete shear walls	6	2½	5	NL	NL	160	100	100
Ordinary reinforced concrete shear walls	5½	2½	4½	NL	NL	NP	NP	NP
Ordinary reinforced masonry shear walls	3	3	2½	NL	160	NP	NP	NP
Intermediate reinforced masonry shear walls	5	3	4½	NL	NL	160	NP	NP
Composite concentrically braced frames	5	2½	4½	NL	NL	160	100	NP
Ordinary composite braced frames	4	2½	3	NL	NL	NP	NP	NP
Ordinary composite reinforced concrete shear walls with steel elements	5	3	4½	NL	NL	NP	NP	NP

TABLE 5.9 Design Coefficients and Factors for Basic Seismic-Force-Resisting Systems (*Continued*)

Basic seismic-force-resisting system	Response modification coefficient, R	System over-strength factor, Ω_o	Deflection amplification factor, C_d	System limitations and building height limitations (ft) by seismic design category				
				A and B	C	D	E	F
Dual systems with intermediate moment frames								
Shear Wall-Frame Interactive System with Ordinary Reinforced Concrete Moment Frames and Ordinary Reinforced Concrete Shear Walls	5½	2½	5	NL	NP	NP	NP	NP
Inverted pendulum systems								
Special steel moment frames	2½	2	2½	NL	NL	NL	NL	NL
Ordinary steel moment frames	1¼	2	2½	NL	NL	NP	NP	NP
Special reinforced concrete moment frames	2½	2	1¼	NL	NL	NL	NL	NL
Structural Steel Systems Not Specifically Detailed for Seismic Resistance	3	3	3	NL	NL	NP	NP	NP

NL indicates not limited.
NP indicates not permitted.

TABLE 5.10 Seismic Design Category Based on Short Period Response Accelerations

Value of S_{DS}	Seismic Use Group I	Seismic Use Group II	Seismic Use Group III
$S_{DS} < 0.167g$	A	A	A
$0.167g \leq S_{DS} < 0.33g$	B	B	C
$0.33g \leq S_{DS} < 0.50g$	C	C	D
$0.50g \leq S_{DS}$	D	D	D

TABLE 5.11 Seismic Design Category Based on 1 Second Period Response Acceleration

Value of S_{DI}	Seismic Use Group I	Seismic Use Group II	Seismic Use Group III
$S_{DI} < 0.067g$	A	A	A
$0.067g \leq S_{DI} < 0.133g$	B	B	C
$0.133g \leq S_{DI} < 0.20g$	C	C	D
$0.20g \leq S_{DI}$	D	D	D

Where E_m equals the earthquake force where seismic forces and dead loads counteract.

- All parts of the structure between separation joints shall be interconnected, and the connections shall be capable of transmitting the seismic force induced in the connection by the parts being connected. Any smaller portion of the structure shall be tied to the remainder of the structure with 5% the weight of the smaller portion. A positive connection for resisting horizontal forces acting on the member shall be provided for each beam, girder, or truss to its support. The connection shall have strength sufficient to resist 5% of the dead and live load vertical reaction applied horizontally.

Analysis Procedures for Seismic Design Categories B, C, D, E, and F. For Seismic Design Categories B and C, IBC 2000 proposed equivalent lateral-load force procedure shall be used. A more rigorous analysis is permitted, too. However, for Seismic Design Categories D, E, and F, the analysis procedures are identified in Table 5.14.

5.19.2 Design Spectral Response Accelerations

Ground motion accelerations, represented by response spectra and coefficients derived from these spectra, shall be determined in accordance with the general procedure or the site-specific procedure. The later procedure shall be used for structures on sites classified as Site Class F.

General Procedure for Determining Maximum Considered Earthquake and Design Spectral Response Accelerations. The maximum considered earthquake spectral response accelerations maps only provide values for Site Class B at short

TABLE 5.12 Plan Structural Irregularities

Irregularities	Irregularity type and description	Seismic Design Category application
1a	Torsional irregularity—to be considered when diaphragms are not flexible	D, E, and F
	Torsional irregularity shall be considered to exist when the maximum story drift, computed including accidental torsion, at one end of the structure transverse to an axis is more than 1.2 times the average of the story drifts at the two ends of the structure	C, D, E, and F
1b	Extreme torsional irregularity—to be considered when diaphragms are not flexible	D
	Extreme torsional irregularity shall be considered to exist when the maximum story drift, computed including accidental torsion, at one end of the structure transverse to an axis is more than 1.4 time the average of the story drifts at the two ends of the structure.	C and D (this irregularity not permitted in E or F)
2	Re-entrant corners	D
	Plan configurations of a structure and its lateral force-resisting system contain re-entrant corners, where both projections of the structure beyond a re-entrant corner are greater than 15% of the plan dimension of the structure in the given direction.	E and F
3	Diaphragm discontinuity	D, E, and F
	Diaphragm with abrupt discontinuities or variations in stiffness, including those having cutout or open areas greater than 50% of the gross enclosed diaphragm area, or changes in effective diaphragm stiffness of more than 50% from one story to the next.	D, E and F
4	Out-of-plane offsets	B, C, and D
	Discontinuities in a lateral force resistance path, such as out-of-plane offsets of the vertical elements.	E and F

TABLE 5.13 Vertical Structural Irregularities

Irregularities	Irregularity type and description	Seismic Design Category application
5	Nonparallel systems The vertical lateral force-resisting elements are not parallel to or symmetric about the major orthogonal axes of the lateral force-resisting system.	C, D, E, and F
1a	Stiffness irregularity—soft story A soft story is one in which the lateral stiffness is less than 70% of that in the story above or less than 80% of the average stiffness of the three stories above.	D, E, and F
1b	Stiffness irregularity—extreme soft story An extreme soft story is one in which the lateral stiffness is less than 60% of that in the story above or less than 70% of the average stiffness of the three stories above.	D This irregularity not permitted in E or F
2	Weight (mass) irregularity Mass irregularity shall be considered to exist where the effective mass of any story is more than 150% of the effective mass of an adjacent story. A roof is lighter than the floor below need not be considered.	D, E and F
3	Vertical geometric irregularity Vertical geometric irregularity shall be considered to exist where the horizontal dimension of the laterl-force-resisting system in any story is more than 130% of that in an adjacent story.	D, E, and F
4	In-plane discontinuity in vertical–lateral-force-resisting elements An in-plane offset of the lateral-force-resisting elements greater than the length of those elements or a reduction in stiffness of the resisting element in the story below.	B, C, D, E and F
5	Discontinuity in capacity-weak story A weak story is one in which the story lateral strength is less than 80% of that in the story above. The story strength is the total strength of seismic-resisting elements sharing the story shear for the direction under consideration.	B, C, and D This irregularity not permitted in E or F

TABLE 5.14 Analysis Procedures for Seismic Design Categories D, E, and F

Structure description	Minimum allowance analysis procedure for seismic design
1. Seismic Use Group—1 building of light framed construction 3 stories or less in height and of other construction, 2 stories or less in height.	Simplified procedure
2. Regular structures other than those in Item 1 above, up to 240 ft/in height.	Equivalent lateral force procedure
3. Structures that have vertical irregularities of type 1a, 1b, 2, or 3 in Table 5.13, or plan irregularities of type 1a or 1b of Table 5.12, and have a height exceeding 5 stories or 65 ft and structures exceeding 240 ft in height.	Model analysis procedure
4. Other structures designated as having plan or vertical irregularities	Equivalent lateral force procedure with dynamic characteristics included in the analytical model
5. Structures with all of the following characteristics: • located in an area with S_{D1} of 0.2 or greater • located in an area assigned to Site Class E or F • with a natural period T of 0.7 seconds or greater, as determined in equivalent lateral force procedure	Model analysis procedure. A site-specific response spectrum shall be used but the design base shear shall not be less than that determined from simplified procedure

period (S_S) and at 1-second period (S_1) and they need to be adjusted for site class effects, by site coefficient F_a and F_v. (See Tables 5.15 and 5.16.)

The corresponding design spectral response accelerations at short periods and at 1 second are:

$$S_{DS} = \frac{2}{3} F_a S_a \tag{5.309}$$

$$S_{D1} = \frac{2}{3} F_v S_1 \tag{5.310}$$

The general design response spectrum curve is developed as Fig. 5.113, in which

$$T_0 = 0.2 \frac{S_{D1}}{S_{DS}}$$

$$T_S = \frac{S_{D1}}{S_{DS}}$$

and T is the fundamental period (in seconds) of the structure.

TABLE 5.15 Values of Site Coefficient F_a as a Function of Site Class and Mapped Spectral Response Acceleration at Short Periods (S_S)

Site class	Mapped spectral response acceleration at short periods				
	$S_S \leq 0.25$	$S_S = 0.50$	$S_S = 0.75$	$S_S = 1.00$	$S_S \geq 1.25$
A	0.8	0.8	0.8	0.8	0.8
B	1.0	1.0	1.0	1.0	1.0
C	1.2	1.2	1.1	1.0	1.0
D	1.6	1.4	1.2	1.1	1.0
E	2.5	1.7	1.2	0.9	[a]
F	Note [a]	Note [a]	Note [a]	Note [a]	Note [a]

[a] Site specific geotechnical investigation and dynamic site response analyses shall be performed to determine appropriate values.

Note: Use straight-line interpolation for intermediate values of mapped spectral acceleration at short period, S_S.

TABLE 5.16 Values of Site Coefficient F_v as a Function of Site Class and Mapped Spectral Response Acceleration at 1-Second Periods (S_1)

Site class	Mapped spectral response acceleration at 1-second period				
	$S_1 \leq 0.1$	$S_1 = 0.2$	$S_1 = 0.3$	$S_1 = 0.4$	$S_1 = 0.5$
A	0.8	0.8	0.8	0.8	0.8
B	1.0	1.0	1.0	1.0	1.0
C	1.7	1.6	1.5	1.4	1.3
D	2.4	2.0	1.8	1.6	1.5
E	3.5	3.2	2.8	2.4	[a]
F	Note [a]	Note [a]	Note[a]	Note [a]	Note [a]

[a] Site specific geotechnical investigation and dynamic site response analyses shall be performed to determine appropriate values.

Note: Use straight-line interpolation for intermediate values of mapped spectral acceleration at 1-second period, S_1.

Site Specific Procedures for Determining Design Spectral Response Accelerations

- A site specific study shall account for the regional seismicity and geology; the expected recurrence rates and maximum magnitudes of events on known faults and source zones; the location of the site with respect to these; near source effects, if any; and the characteristics of subsurface site conditions.
- The probabilistic maximum considered earthquake ground motion shall be taken as that motion represented by an acceleration response spectrum having a 2% probability of exceedance within a 50-year period. The probabilistic maximum considered earthquake spectral response acceleration at any period, S_{aM}, shall be taken from the 2% probability of exceedance within a 50-year period spectrum (where S_{aM} exceeds the deterministic limit shown in Fig. 5.114.)

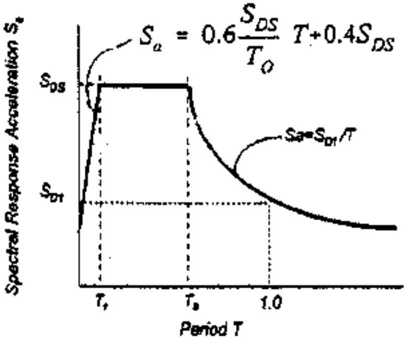


FIGURE 5.113 Design Response Spectrum

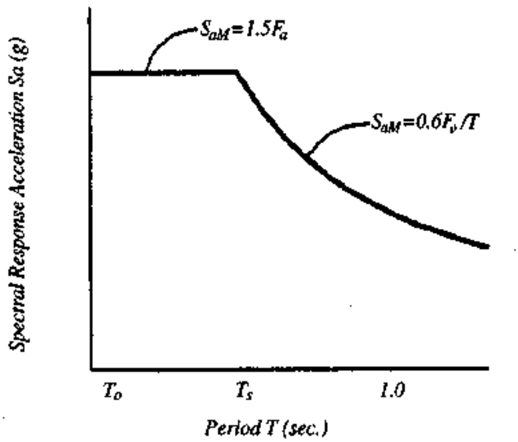


FIGURE 5.114 Deterministic Limit on Maximum Considered Earthquake Response Spectrum

- The maximum considered earthquake ground motion spectrum shall be taken as the lesser of the probabilistic maximum considered earthquake ground motion or the deterministic maximum considered earthquake ground motion spectrum S', but shall not be taken as less than the deterministic limit ground motion as shown in Fig. 5.114. S′ is calculated as 150% of the median spectral response accelerations (S_{aM}) at all periods resulting from a characteristic earthquake on any known active fault within the region.

The site-specific design spectral response acceleration S_a at any period can be expressed as

$$S_a = \frac{2}{3} S_{aM} \tag{5.311}$$

- S_a shall be no less than 80% of the corresponding value as the general design response on Fig. 5.113.
- The design spectral response acceleration coefficients at short periods, S_{DS} and the design spectral response acceleration at 1-second period, S_{D1}, shall be taken the values S_a at periods of 0.2 second and 1.0 second, respectively.

5.19.3 Minimum Design Lateral Force and Related Effects

From Table 5.14, we know that there are several seismic force analysis procedures, such as simplified procedure, equivalent lateral force procedure, model analysis procedure. The reader should note that another method, the dynamic analysis procedure, is not presented here. Different Seismic Design Categories require different analysis procedures. Among these analysis procedures, the equivalent lateral force procedure is the most popular approach because of its easy calculation and clear seismic design concepts. It can also be used as the preliminary design seismic force for the Seismic Design Categories that require more rigorous analysis procedures. In this handbook, we will only cover this analysis procedure.

Equivalent Lateral Force Procedure. In this analysis, a building is considered to be fixed at the base. The seismic base shear, which is equivalent to the total horizontal forces at the base generated by a seismic force in any direction, can be expressed as

$$V = C_S W \tag{5.312}$$

where C_S is the response coefficient and W is the effective seismic weight of the structure, including the total dead load and other loads listed below:

1. In areas used for storage, a minimum of 25% of the reduced floor live load (floor live load in public garages and open parking structures need not be included)
2. Where an allowance for partition load is included in the floor load design, the actual partition weight or a minimum weight of 10 psf of floor area (whichever is greater)
3. Total operating weight of permanent equipment
4. 20% of flat roof snow load where the flat roof snow load exceeds 30 psf

The seismic response coefficient, C_S, shall be determined in accordance with the following formula:

$$C_S = \frac{S_{DS}}{\left(\dfrac{R}{I}\right)} \tag{5.313}$$

where S_{DS} = the design spectral response acceleration at short period
R = the response modification factor from Table 5.9
I = the Occupancy Importance Factor

The value of the seismic response coefficient C_S need not exceed the following:

$$C_S = \frac{S_{DI}}{\left(\frac{R}{I}\right) T} \tag{5.314}$$

but shall not be taken less than:

$$C_S = 0.44 S_{D1} I \tag{5.315}$$

For buildings and structures in Seismic Design Categories E or F, and those buildings and structures for which the 1-second spectral response S_1 is equal to or greater than 0.6g, the value of the seismic response coefficient C_S shall not be taken as less than:

$$C_S = \frac{0.5 S_1}{R/I} \tag{5.316}$$

where I and R are defined above and

S_{D1} = the design spectral response acceleration at 1-second period
T = the fundamental period of the building (seconds)
S_1 = the maximum considered earthquake spectral response acceleration at 1-second period

The fundamental period of the building, T in the direction under consideration shall be established using the structural properties and deformational characteristics of the resisting elements in a properly substantiated analysis, or shall be taken as the approximate fundamental period T_a. The calculated fundamental period T shall not exceed the product of the coefficient for the upper limit on the calculated period C_u, from Table 5.17, and the approximate fundamental period T_a. The approximate fundamental T_a shall be determined as:

$$T_a = C_T h_n^{3/4} \tag{5.317}$$

TABLE 5.17 Coefficient for Upper Limit on Calculated Period

Design spectral response acceleration at 1-second period, S_{D1}	Coefficient C_u
≥0.4	1.2
0.3	1.3
0.2	1.4
0.15	1.5
≤0.1	1.7

where C_T = building period coefficient (see following list of coefficient values)

- 0.035 for moment resisting frame systems of steel in which the frames resist 100% of the required seismic force and are not enclosed or adjoined by more rigid components that will prevent the frames from deflecting when subjected to seismic forces,
- 0.030 for moments resisting frame systems of reinforced concrete in which the frames resist 100% of the required seismic force and are not enclosed or adjoined by more rigid components that will prevent the frames from deflecting when subjected to seismic forces,
- 0.030 for eccentrically braced steel frames,
- 0.020 for all other building systems.

h_n = the height (ft) above the base to the highest level of the building.

Alternately, determination of the approximate fundamental period T_a in seconds, from the following formula for concrete and steel moment-resisting frame buildings not exceeding 12 stories in height and having a minimum story height of 10 ft, is permitted:

$$T_a = 0.1N \tag{5.318}$$

where N is the number of stories.

The base shear V is distributed vertically to the n stories as lateral forces F:

$$F_x = C_{vx}V \tag{5.319}$$

$$C_{vx} = \frac{w_x h_x^k}{\sum_{i=1}^{n} w_i h_i^k} \tag{5.320}$$

where C_{vx} = vertical distribution factor,
w_i and w_x = the portion of the total gravity load of the building, W, located or assigned to level i or x,
h_i and h_x = the height (ft) from the base to level i or x, and
k = a distribution exponent related to the building period as follows:

- For buildings having a period of 0.5 seconds or less, $k = 1$
- For buildings having a period of 2.5 seconds or more, $k = 2$
- For buildings having a period between 0.5 and 2.5 seconds, k shall be 2 or shall be determined by linear interpolation between 1 and 2.

The seismic design story shear in any story, V_x is

$$V_x = \sum_{i=1}^{n} F_i \tag{5.321}$$

Rigid Diaphragms. For rigid diaphragms the seismic design story share, V_x shall be distributed to the various vertical elements of the seismic force-resisting system in the story under consideration based on the relative lateral stiffness of the vertical force resisting elements and the diaphragm.

For flexible diaphragms, seismic design story shear, V_x shall be distributed to various vertical elements based on the tributary area of the diaphragm to each line of resistance. For the purpose of this section, the vertical elements of the lateral force-resisting system are permitted to be considered to be in the same line of resistance, if the maximum out-of-plane offset between each of the elements is less than 5% of the building dimension perpendicular to the direction of lateral load.

Torsion. Where diaphragms are not flexible, the design shall include the torsional moment M_t, resulting from the difference in locations of the center of mass and the center of stiffness. Also where diaphragms are not flexible, in addition to the torsional moment, the design shall include accidental torsional moments M_{ta}, caused by assumed displacement of the center of mass, each way from its actual location, by a distance equal to 5% of the dimension of the building perpendicular to the direction of the applied forces.

Dynamic Amplification of Torsion. For a structure in Seismic Design Category C, D, E, or F, where Type 1a or 1b plan torsional irregularity exists, effects of torsional irregularity shall be accounted for by multiplying the sum of M_t plus M_{ta} at each level by a torsional amplification factor, A_x, determined from the following formula:

$$A_x = \left(\frac{\delta_{\max}}{1.2\delta_{\mathrm{avg}}}\right)^2 \tag{5.322}$$

where $\delta_{\max}$ = the maximum displacement at level x and
δ_{avg} = the average of the displacement at the extreme points of the structure at level x

The torsional amplification factor, A_x, is not required to exceed 3.0. The more severe loading for each element shall be considered for design.

Overturning. The building shall be designed to resist overturning effects caused by the seismic forces. At any story, the increment of overturning moment in the story under consideration shall be distributed to the various vertical force-resisting elements in the same proportion as the distribution of the horizontal shears to those elements.

The overturning moments at level x, M_x shall be determined from the following formula:

$$M_x = \tau \sum_{i=x}^{n} F_i(h_i - h_x) \tag{5.323}$$

where F_i = the portion of the seismic base shear V, induced at Level i,
h_i and h_x = the height from the base to level i or x,
τ = the Overturning Moment Reduction Factor

The Overturning Moment Reduction Factors are

- 1.0 for the top 10 stories,
- 0.8 for the 20th story from the top and below, and
- value between 1.0 and 0.8 determined by a straight line interpolation for stories between the 20th and 10th stories below the top.

Story Drift Determination. The design story drift Δ shall be computed as the difference of the deflections at the center of mass at the top and bottom of the story under consideration. Where allowable stress design is used. Δ shall be computed using earthquake forces without dividing by 1.4. For structures assigned to Seismic Design Category C, D, E, or F having plan irregularity types 1a or 1b of Table 5.12, the design story drift Δ shall be computed as the largest difference of the deflections along any of the edges of the structure at the top and bottom of the story under consideration.

The deflections of level x, δ_x, shall be determined in accordance with following formula:

$$\delta_x = \frac{C_d \delta_{xe}}{I} \tag{5.324}$$

where C_d = the deflection amplification factor in Table 5.9,
δ_{xe} = the deflections determined by an elastic analysis of the seismic force resisting system, and
I = the Occupancy Importance Factor

For purposes of this drift analysis only, the upper bound limitation specified on the computed fundamental period, T, in seconds, of the building, shall not apply.

The design story drift Δ shall be increased by the incremental factor relating to the P-delta effects. When calculating drift, the redundancy coefficient ρ shall be taken as 1.0.

P-Delta Effects. P-delta effects on story shears and moments the resulting member forces and moments, and the story drifts induced by these effects are not required to be considered when the stability coefficient θ, as determined by the following formula, is equal to or less than 0.10:

$$\theta = \frac{P_x \Delta}{V_x h_{sx} C_d} \tag{5.325}$$

where P_x = the total unfactored vertical design load at and above level x; when calculating the vertical design load for purposes of determining P-delta, the individual load factors need not exceed 1,0;
Δ = the design story drift occurring simultaneously within V_x;
V_x = the seismic shear force acting between level x and $x - 1$;
h_{sx} = the story height below level x; and
C_d = the deflection amplification factor in Table 5.9

The stability coefficient θ shall not exceed θ_{max} determined as follows:

$$\theta_{max} = \frac{0.5}{\beta C_d} \leq 0.25 \tag{5.326}$$

where β is the ratio of shear demand to shear capacity for the story between level x and $x - 1$. Where the ratio β is not calculated, a value of $\beta = 1.0$ shall be used.

When the stability coefficient θ is greater than 0.10 but less than or equal to θ_{max}, interstory drifts and element forces shall be computed including P-delta effects. To obtain the story drift for including the P-delta effect, the design story drift shall be multiplied by $1.0/(1 - \theta)$. Where θ is greater than θ_{max} the structure is potentially unstable and shall be redesigned.

Seismic Load Effect. Where the effects of gravity and seismic loads are additive, seismic load E shall be defined as:

$$E = \rho Q_E + 0.2 S_{DS} D \tag{5.327}$$

and where the effects of gravity counteract the seismic load, seismic load E shall be defined as

$$E = \rho Q_E - 0.2 S_{DS} D \tag{5.328}$$

where E = the combined effect of horizontal and vertical earthquake-induced forces,
ρ = a reliability factor based on system redundancy,
Q_E = the effect of horizontal seismic forces,
S_{DS} = the design spectral response acceleration at short periods,
D = the effect of dead load

Where seismic forces and dead loads are additive,

$$E_m = \Omega_o Q_E + 0.2 S_{DS} D \tag{5.329}$$

Where seismic forces and dead loads counteract,

$$E_m = \Omega_o Q_E - 0.2 S_{DS} D \tag{5.330}$$

Where E, Q_E, S_{DS} and D are as defined above and Ω_o is the system overstrength factor as given in Table 5.9. The terms $\Omega_o Q_E$ need not exceed the maximum force that can be transferred to the element by the other elements of the lateral force-resisting system.

Where allowable stress design methodologies are used with the special load combinations with E_m design strengths are permitted to be determined using an allowable stress increase of 1.7 and a resistance factor ϕ, of 1.0.

Redundancy. A redundancy coefficient, ρ, shall be assigned to all structures in accordance with this section, based on the extent of structural redundancy inherent in the lateral forces resisting system.

For structure assigned to Seismic Design Category A, B, or C, the value of the redundancy coefficient ρ is 1.0. For structures in Seismic Design Categories D, E, and F, the redundancy coefficient ρ shall be taken as the largest of the values of ρ_i, calculated at each story i of the structure as follows:

$$\rho_i = 2 - \frac{20}{r_{max_i} \sqrt{A_i}} \tag{5.331}$$

where r_{max} = the ratio of the design shear resisted by the most heavily loaded single element in the story to the total story shear, for a given direction of loading

- For braced frames the value of r_{max_i} is equal to the lateral force component in the most heavily loaded braced element divided by the story shear.
- For moment frames, r_{max_i} shall be taken as the maximum of the sum of the shears in any two adjacent columns in a moment frame divided by the story shear. For columns common to two bays with moment resisting connections on opposite sides at the level under consideration, it is permitted to use 70% of the shear in that column in the column shear summation.
- For shear walls, r_{max_i} shall be taken as the maximum value of the product of the shear in the wall or wall pier and $10/\ell_W$, divided by the story shear, where ℓ_W is the length of the wall or wall pier in feet.
- For dual systems, r_{max_i} shall be taken as the maximum value defined above, considering all lateral load-resisting elements in the story. The lateral loads shall be distributed to elements based on relative rigidities considering the interaction of the dual system. For dual systems, the value of ρ need not exceed 80% of the value calculated above.

A_i = the floor area in square feet of the diaphragm level immediately above the story.

The value of ρ shall not be less than 1.0, and need not exceed 1.5.

For structures with seismic force resisting systems in any direction comprised solely of special moment frames, the seismic force-resisting system shall be configured such that the value of ρ calculated in accordance with this section does not exceed 1.25 for structures assigned to Seismic Design Category D, and does not exceed 1.1 for structures assigned to Seismic Design Category E or F.

Deflections and Drift Limits. The design story drift Δ shall not exceed the allowable story drift Δ_a, as obtained from Table 5.18 for any story. All portions of the building shall be designed to act as an integral unit in resisting seismic forces unless separated structurally by a distance sufficient to avoid damaging contact under total deflection δ_x.

5.19.4 Design Detailing Requirements and Structural Component Load Effects

In order to provide a more reliable and consistent level of seismic safety in new building construction, IBC 2000 includes a much larger set of provisions on proportioning and detailing structural members and system. The Code requirements are based on Seismic Design Category. These special requirements are for items such as openings in shear walls and diaphragms, diaphragm design, collector element design, design of bearing walls and shear walls and their anchorage, direction

TABLE 5.18 Allowable Story Drift, Δ_a[a]

Building	Seismic use group I	Seismic use group II	Seismic use group III
Building, other than masonry shear wall or masonry wall frames buildings, four stories or less in height with interior walls, partitions, ceilings, and exterior wall systems that have been designed to accommodate the story drifts	$0.025\ h_{sx}$[b]	$0.020\ h_{sx}$	$0.015\ h_{sx}$
Masonry cantilever shear wall buildings[c]	$0.010\ h_{sx}$	$0.010\ h_{sx}$	$0.010\ \mathrm{h}_{xx}$
Other masonry shear wall buildings	$0.007\ h_{xx}$	$0.007\ h_{sx}$	$0.007\ h_{sx}$
Masonry wall frame buildings	$0.013\ h_{sx}$	$0.013\ h_{sx}$	$0.010\ h_{sx}$
All other buildings	$0.020\ h_{sx}$	$0.015\ h_{sx}$	$0.010\ h_{sx}$

[a] There shall be no drift for single-story buildings with interior walls, partitions, ceilings, and exterior wall systems that have been designed to accommodate the story drifts.
[b] h_{sx} is the story height below Level x.
[c] Buildings in which the basic structural system consists of masonry shear walls designed as vertical elements cantilevered from their base or foundation support which are so constructed that moment transfer between shear walls coupling is negligible.

of seismic load impact, and so on. It is very important that the design engineer be familiar with requirements.

5.19.5 Seismic Design Requirements on Nonstructural Components

Architectural, mechanical, electrical, and other nonstructural components in structures shall be designed and constructed to resist equivalent static forces and displacements. Unless otherwise noted, components shall be considered to have the same Seismic Design Category as the structure that they occupy or to which they are attached.

The interrelationship of components and their effect on each other shall be considered so that the failure of any essential or nonessential architectural, mechanical, or electrical component shall not cause the failure of another essential architectural, mechanical, or electrical component.

Component Force Transfer. The component shall be attached such that the component forces are transferred to the structure of the building. Component seismic attachments shall be bolted, welded, or otherwise positively fastened without consideration of frictional resistance produced by the effects of gravity.
The Seismic Force F_p is

$$F_p = \frac{0.4\alpha_p S_{\mathrm{DS}} W_p}{\left(\dfrac{R_p}{I_p}\right)}\left(1 + 2\,\frac{z}{h}\right) \tag{5.332}$$

and

$$0.3S_{\mathrm{DS}}I_pW_p \leq F_p \leq 1.6S_{\mathrm{DS}}I_pW_p$$

where F_p = Seismic design force centered at the component's center of gravity and distributed relative to component's mass distribution
S_{DS} = Design spectral response acceleration at short period
α_p = Component amplification factor that varies from 1.00 to 2.50 (select appropriate value from Tables 5.19 and 5.20)
I_p = Component importance factor that is 1.5 for life safety component and 1.0 for all other components
W_p = Component operating weight
R_p = Component response modification factor that varies from 1.0 to 5.0 (select appropriate value from Tables 5.19 and 5.20)
z = Height in structure at point of attachment of component. For items at or below the base, z shall be taken as 0.
h = Average roof height of structure relative to the base elevation.

The force F_p shall be applied independently longitudinally and laterally in combination with service loads associated with the component. Component earthquake effects shall be determined for combined horizontal and vertical load effects as Q_{E} in E. The redundancy based reliability coefficient, p, is permitted to be taken as equal to 1.

(J. M. Biggs, "Introduction to Structural Dynamics," and R. Clough and J. Penzien, "Dynamics of Structures," McGraw-Hill Publishing Company, New York; E. Rosenblueth, "Design of Earthquake-Resistant Structures," Halsted/Wiley, Somerset, N.J.; N. M. Newmark and E. Rosenblueth, "Fundamentals of Earthquake Engineering," Prentice-Hall, Englewood Cliffs, N.J.; S. Okamoto, "Introduction to Earthquake Engineering," John Wiley & Sons, Inc., New York, International Building Code 2000)

5.20 FLOOR VIBRATIONS

Excessive vibration can be characterized as too large for sensitive equipment or too large for occupant comfort. Determining these permissible levels is an entire research area in itself; however, some of the more widely accepted levels are discussed in following paragraphs. These levels are expressed by researchers in terms of either acceleration, velocity, or displacement amplitudes and are often frequency-dependent. There is no consensus as to the most relevant measure for describing acceptable levels.

Comfort of the occupants is a function of human perception. This perception is affected by factors including the task or activity of the perceiver, the remoteness of the source, and the movement of other objects in the surroundings. A person is distracted by acceleration levels as small as 0.5% g. Multiple-use occupancies must therefore be carefully considered.

Webster and Vaicitis describe a facility that has both dining and dancing in a large open area. The floor was noted to have a first natural frequency of 2.4 Hz, which is in resonance with the beat of many popular dance song. This resonance response produced maximum acceleration and displacement levels of 7% g and 0.13 in, respectively. Such levels actually caused sloshing waves in drinks and noticeable bouncing of the chandeliers. The occupants found these levels to be quite objectionable.

TABLE 5.19 Architectural Components Coefficients

Architectural component or element	α_p[a]	R_p
Interior nonstructural walls and partitions		
Plain (unreinforced) masonry walls	1.0	1.25
Other walls and partitions	1.0	2.5
Cantilever elements (unbraced or braced to structural frame below its center of mass)		
Parapets and cantilever interior nonstructural walls	2.5	2.5
Chimneys and stacks when laterally braced or supported by the structural frame	2.5	2.5
Cantilever elements (braced to structural frame above its center of mass)		
Parapets	1.0	2.5
Chimneys and Stacks	1.0	2.5
Exterior Nonstructural Walls	1.0	2.5
Exterior nonstructural wall elements and connections		
Wall element	1.0	2.5
Body of wall panel connections	1.0	2.5
Fasteners of the connecting system	1.25	1.0
Veneer		
Limited deformity elements and attachments	1.0	2.5
Low deformity elements or attachments	1.0	1.25
Penthouse (except when framed by an extension of the building frame)	2.5	3.5
Ceilings	1.0	2.5
Cabinets		
Storage cabinets and laboratory equipment	1.0	2.5
Access floors		
Special access floors	1.0	2.5
All other	1.0	1.25
Appendages and ornamentations	2.5	2.5
Signs and billboards	2.5	2.5
Other rigid components		
High deformability elements and attachments	1.0	3.5
Limited deformability elements and attachments	1.0	2.5
Low deformability materials and attachments	1.0	1.25
Other flexible components		
High deformability elements and attachments	2.5	3.5
Limited deformability elements and attachments	2.5	2.5
Low deformability materials and attachments	2.5	1.25

[a] Where justified by detailed analyses, a lower value for α_p is permitted, but shall not be less than 1. The reduced value of α_p shall be between 2.5, assigned to flexible or flexibly attached equipment, and 1, assigned to rigid or rigidly attached equipment.

Many different scales and criteria are available which address the subjective evaluation of floor vibration. Factors included in these subjective evaluations include the natural frequency of the floor system, the maximum dynamic amplitude (acceleration, velocity, or displacement) due to certain excitations, and the amount of damping present in the floor system. At the present time, most of the design criteria utilize either a single impact function to assess vibrations, which are tran-

TABLE 5.20 Mechanical and Electrical Components Coefficients

Mechanical and electrical component or element	α_p[a]	R_p
General mechanical		
Boilers and furnaces	1.0	2.5
Pressure vessels on skirts and free-standing	2.5	2.5
Stacks	2.5	2.5
Cantilevered chimneys	2.5	2.5
Other	1.0	2.5
Manufacturing and process machinery		
General	1.0	2.5
Conveyors (nonpersonnel)	2.5	2.5
Piping systems		
High deformability elements and attachments	1.0	3.5
Limited deformability elements and attachments	1.0	2.5
Low deformability elements or attachments	1.0	1.25
HVAC system equipment		
Vibration isolated	2.5	2.5
Nonvibration isolated	1.0	2.5
Mounted in-line with ductwork	1.0	2.5
Other	1.0	2.5
Elevator components	1.0	2.5
Escalator component	1.0	2.5
Trussed towers (free-standing or guyed)	2.5	2.5
General electrical		
Distributed systems (bus ducts, conduit, cable tray)	2.5	5.0
Equipment	1.0	2.5
Lighting Fixtures	1.0	1.25

[a] Where justified by detailed analyses, a lower value for α_p is permitted, but shall not be less than 1. The reduced value of α_p shall be between 2.5, assigned to flexible or flexibly attached equipment, and 1, assigned to rigid or rigidly attached equipment.

sient in nature, or a sinusoidal function to assess steady-state vibrations from rhythmic activities.

5.21 WISS AND PARMELEE RATING FACTOR FOR TRANSIENT VIBRATIONS

Wiss and Parmelee also conducted research to refine the findings of Lenzen's research. In particular, they attempted to quantify, in a more scientifically rigorous manner, human perception to transient floor motion. They subjected 40 persons, standing on a vibrating platform, to transient vibration episodes with different combinations of frequency (2.5 to 25 Hz), peak displacements (0.0001 to 0.10 in), and damping (0.1 to 0.16, expressed as a ratio of critical). After each episode, the subject was asked to rate the vibration on a scale of 1 to 5 with the following definitions: (1) imperceptible, (2) barely perceptible, (3) distinctly perceptible, (4) strongly perceptible, and (5) severe. Using regression analysis, an equation was developed which related the three variables of the vibration episode to the subjective perception ratings. This equation is presented below.

Wiss and Parmelee rating factor:

$$R = 5.08 \left(\frac{FA}{D^{0.217}} \right)^{0.265}$$

where R = response rating; 1 = imperceptible; 2 = barely perceptible; 3 = distinctly perceptible; 4 = strongly perceptible; 5 = severe.
F = frequency of the vibration episode, Hz
A = maximum displacement amplitude, in
D = damping ratio, expressed as a ratio of critical

A graph of this subjective rating system is shown in Fig. 5.115. It should be noted that the lines represent a mean for that particular rating. The authors suggest that the boundaries for each rating lie halfway between the mean lines. The boundaries defining R = 1 and R = 5 are not identified by the authors. These ratings are unbounded; therefore, a mean line cannot be computed.

5.22 REIHER-MEISTER SCALE FOR STEADY-STATE VIBRATIONS

The scale discussed below and those in Art. 5.21 and 5.23 useful in assessing human perception to vibration levels. They are presented to provide insight with respect

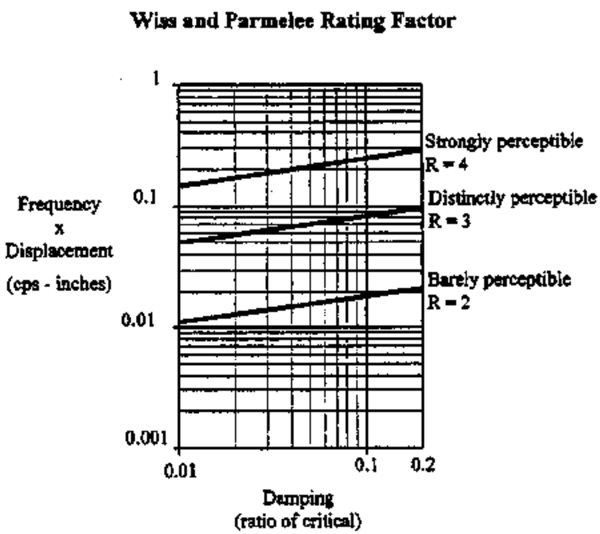


FIGURE 5.115 Wiss and Parmelee rating factor scale.[17]

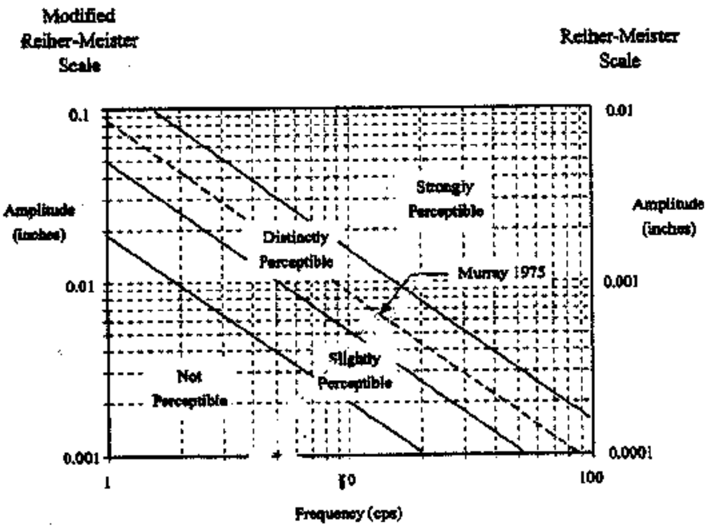


FIGURE 5.116 Modified Reiher-Meister and Reiher-Meister scales.

TABLE 5.21 Estimates of Floor System Damping

Component	Damping (% of critical)	Description
Bare floor	1–3%	Lower limit for thin slab of lightweight concrete; upper limit for thick slab of normal weight concrete
Ceiling	1–3%	Lower limit for hung ceiling; upper limit for sheetrock on furring attached to beams of joists
Ductwork and mechanial	1–10%	Depends on amount and attachment
Partition	10–20%	If attached to the floor system and not spaced more than every five floor beams of the effective joist floor width

(Serviceability Considerations for floors and roof systems Chapter 9, "Steel Design Handbook" by Akbar Tamboli, McGraw-Hill Book Company, New York.)

to the vibration levels which annoy occupants as well as a historical perspective on the development of floor vibration criteria. Reiher and Meister[15] published a frequently referenced scale concerning human perception to steady-state vibration. While this scale was not derived specifically for the evaluation of floor systems, it has been extrapolated by other researchers for such purposes. The scale represented by the right-hand axis of the graph in Fig. 5.116 was derived from the subjective evaluations of 10 persons standing on a vibrating platform. The subjects were exposed to vertical steady-state vibration episodes each lasting approximately 5 minutes, and were asked to classify the vibration as (1) slightly perceptible, (2) distinctly perceptible, (3) strongly perceptible, (4) disturbing, and (5) very disturbing. The frequency and displacement ranges of the episodes were 5 to 70 Hz and 0.001 to 0.40 in, respectively.

5.23 MURRAY CRITERION FOR WALKING VIBRATIONS

5.23.1 Summary of the Criterion

In the criterion presented by Murray, an acceptable steel floor system is predicted, with respect to vibration levels due to walking excitation, if the dynamic criterion below is met. This criterion is applicable to offices and residences with fundamental natural frequencies below 10 Hz.

Murray criterion:

$$D > 35A_0 f + 2.5$$

where D = damping in floor system, expressed as a percent of critical
A_0 = maximum initial amplitude of the floor system due to a heel-drop excitation, in
f = first natural frequency of the floor system, Hz

This criterion is only applicable for the units specified. The reader is cautioned against using other units.